ONE DIMENSIONAL MOTION IN A THICK SEXTUPOLE A COMPARISON OF TRACKING METHODS WITH THE EXACT SOLUTION

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1. Introduction

In a recent report [1], the results of various mapping techniques were compared with the exact solution of a nonlinear equation. For this purpose, the equation of motion of a pendulum was chosen, for which the exact solution can be expressed in terms of elliptic functions. In all cases discussed in this report, the relative errors of the solution obtained by rather high-order mapping reached 10% after a few hundred (or at most thousand) periods of oscillation, and seemed to increase exponentially, so a few more periods would give completely wrong results.

Such mappings are used widely for the design and modelling of particle accelerators and storage rings. A limitation to low numbers of periods would therefore have far-reaching consequences, in particular for proton storage rings where - due to the negligible effect of radiation damping - a large number of turns need to be tracked. After verifying those claims, it thus appeared important to us to determine the exact conditions under which such limitations are valid.

A number of factors influence the speed of divergence, such as making the maps energy conserving and/or symplectic, or the distinction between amplitude and phase errors. However, the most important result of our study was the realization of the importance of the discretization or "stepsize" in the independent variable (usually time or distance along the circumference). Not only does the convergence become worse with increasing stepsize as expected, but there exists a limiting step-size above which mappings diverge to all orders. The examples given in the above-mentioned report were all computed for a stepsize of 0.8 periods, well beyond the limiting value for the pendulum (about 0.42 periods for a pendulum starting at 90^0).

While these results explain the observed behaviour, they still limit tracking by Taylor maps in the presence of nonlinearities. A too large stepsize leads invariably to divergence, and hence care must be taken when many nonlinear elements are concatinated. In general, a comparison of results with different stepsizes will show whether the limit has been exceeded.

In this report, we compare the exact solutions for the one-dimensional motion in a sextupole - still expressible by elliptic functions and integrals - with Taylor maps in the stepsize of the independent variable. This permits us to obtain explicit criteria for the convergence of the mapping. We also discuss extension of this model to 2-dimensional motion and to more general multipoles, and the technique of tracking to computer accuracy.

2. Mapping by Taylor-series

The differential equations describing the motion of a charged particle inside a magnetic multipole are nonlinear, autonomous, coupled equations which in general permit no closed form solution. A common technique to obtain approximate solutions is to establish a functional relation between the entrance and exit coordinates and momenta for the given multipole :

$$\vec{X}_1 = \vec{F}\,(\vec{X}_0,\, \Delta s) \qquad (1)$$

where the components of $\vec{F}$ are expressed as functions of the components of $\vec{X}_0$. Here Δs stands for the length of the elements hence $\vec{X}_1 = \vec{X}(\Delta s)$. A common method to create $\vec{F}$ is to expand it into a Taylor-series w.r.t. the entrance coordinates x_0 and $u_0 = dx/ds(0)$. As an example we take the horizontal motion in a sextupole. Neglecting the effect of end fields, the equations for the horizontal motion inside a thick sextupole are :

$$\dot{x} = u$$

$$\dot{u} = -\alpha x^2 \;;\quad \alpha = 2k' \qquad (2)$$

Using the second order "ansatz" :

$$\vec{F}(\vec{X}_0,\Delta s) = \begin{vmatrix} m_1x_0 + m_2u_0 + m_{11}x_0^2 + m_{12}x_0u_0 + m_{22}u_0^2 \\ m_3x_0 + m_4u_0 + m_{33}x_0^2 + m_{34}x_0u_0 + m_{44}u_0^2 \end{vmatrix} \qquad (3)$$

inserting Eq. (3) into Eq. (2), and comparing the coefficients of like powers in x_0 and u_0 we uniquely determine the coefficients m_i and m_{ij} as functions of the sextupole length Δs. The results are :

$$\begin{aligned} &m_1 = m_4 = 1, \quad m_2 = \Delta s, \qquad m_3 = 0 \\ &m_{11} = -\alpha/2\ \Delta s^2,\ m_{12} = -\alpha/3\ \Delta s^3,\ m_{22} = -\alpha/12\ \Delta s^4 \\ &m_{33} = -\alpha\ \Delta s, \quad m_{34} = -\alpha\ \Delta s^2, \quad m_{44} = -\alpha/3\ \Delta s^3 \end{aligned} \qquad (4)$$

This method is used widely in tracking programs and has first been introduced in Ref.[2]. An alternative method has recently been used in Ref.[3] : it consists of expanding the mapping in terms of the element length Δs instead of the entrance coordinates. The advantage of this method is that the expansion coefficients can be found directly by repeated differentiation of the equation of motion. To demonstrate this we rewrite (2) as a single second order equation :

$$\ddot{x} = -\alpha\, x^2 \qquad (5)$$

Differentiating Eq. (5) twice and writing x as a Taylor-series up to the fourth order, we find :

$$x(\Delta s) = x_0 + \Delta s u_0 - \frac{\alpha}{2}\, x_0^2 \Delta s^2 - \frac{\alpha}{3}\, x_0 u_0\ \Delta s^3 - \frac{\alpha}{12}\,(u_0^2 - \alpha x_0^3)\Delta s^4$$

$$u(\Delta s) = u_0 - \alpha\, x_0^2 \Delta s - \alpha\, x_0 u_0\ \Delta s^2 - \frac{\alpha}{3}\,(u_0^2 - \alpha x_0^3)\ \Delta s^3 \qquad (6)$$

By comparing Eq. (6) to Eqs. (3) and (4), we realize that all second order terms found by expanding w.r.t. x_0 and u_0 are also contained in Eq. (6). In addition, Eq. (6) contains some (but not all) third order terms in x_0 and u_0. The described method can easily be extended to very high orders, e.g. all terms up to order 20 have been computed in Ref.[3] by using an algebraic manipulator like REDUCE [5]. To check the quality of this method, we compared the results with the exact solution of Eq. (5) which has been derived, e.g. in Ref.[3] :

$$x\,(s) = \beta \left| 1 - 3^{1/2}\, \frac{1 - \mathrm{cn}(u|m)}{1 + \mathrm{cn}(u|m)} \right| \qquad (7)$$

where $\beta^3 = x_0^3 + 3\ u_0^2/(2\alpha)$ and the argument of the Jacobian elliptic function cn is $u = F(\phi_0|m) - \gamma s$, where $\cos\phi_0 = [\beta(3^{1/2}-1) + x_0]/[\beta(3^{1/2}+1) - x_0]$, $\gamma^2 = 2\alpha\beta/3^{1/2}$, and $m = {}^1/_2 + 3^{1/2}/4$.

In order to make the solution bounded, we enclosed a long sextupole by two short quadrupoles. Such a configuration simulates e.g. sextupolar field components in machines with superconducting dipoles. Figs. 1 and 2 show the phase space distance d (in percent) between the trajectories obtained with the exact solution for sextupoles (Eq.(7)) and a 10th and a 14th order mapping (in Δs). A 10% error is reached after 2000 periods for the 10th order mapping, but only after 64000 for the 14th order.

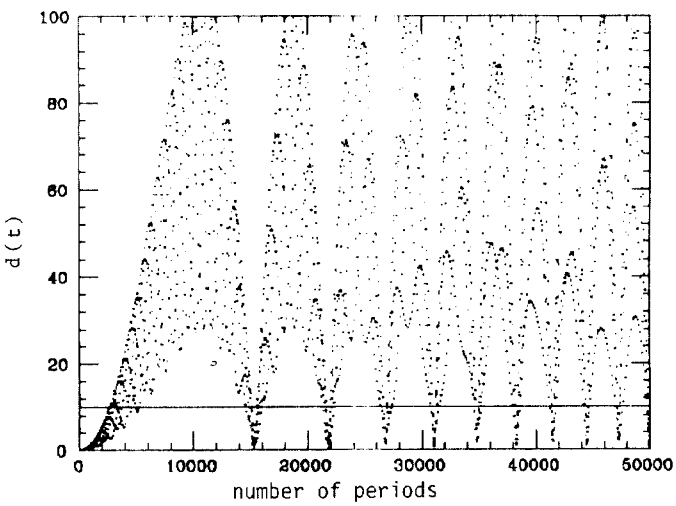

Fig.1: Phase space error d [%] for mapping order N=10.

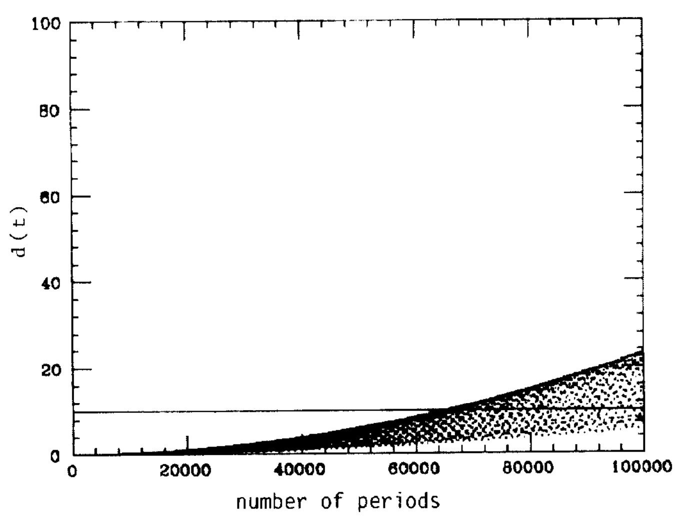

Fig.2: Phase space error d [%] for mapping order N=14.

3. Limitations of Taylor mappings

From the results obtained so far, it seems as if one could obtain any desired accuracy for the solution of nonlinear equations of motion by truncating the series of the mapping at sufficiently high order. However, it is a well known fact that many nonlinear differential equations, including the equations of motion inside thick multipoles, have solutions with poles which depend on the initial conditions [4]. As an illustrative example consider the equation :

$$y'(x) + 2x\, y^2(x) = 0 \qquad (8)$$

which has the solution :

$$y(x) = (x^2 + y_0^{-1})^{-1} \qquad (9)$$

It has a pole at $x = iy_0^{-1/2}$ i.e. depending on y_0. It is a basic property of Taylor-series to diverge if the distance from the point of development to the point of evaluation is larger than that to the closest pole in the complex plane. E.g. if we represent the solution of Eq. (8) by a Taylor-series as :

$$y(x) = \sum_{n=0}^{\infty} \beta_n(y_0)\, x^n \qquad (10)$$

this series will diverge for $|x| > y_0^{-1/2}$.

From the exact solution for the motion inside a thick sextupole (Eq.(7)), we easily detect a pole on the real axis by setting $cn(u|m) = -1$, from which follows the distance to this pole. For $u_0 = 0$ we find :

$$\Delta s = \frac{2\,K(m)}{(\alpha\, x_0)^{1/2}} = \frac{5.16}{(\alpha\, x_0)^{1/2}} \qquad (11)$$

The Taylor series representing the solution of Eq. (5) will diverge if Δs becomes larger than this critical value. We can generalize the limit given by Eq. (11) to the case of arbitrary initial conditions x_0 and u_0. As expected, Δs decreases as either x_0 and u_0 increases. In Fig. 3 we show the limiting Δs as a function of x_0 and u_0.

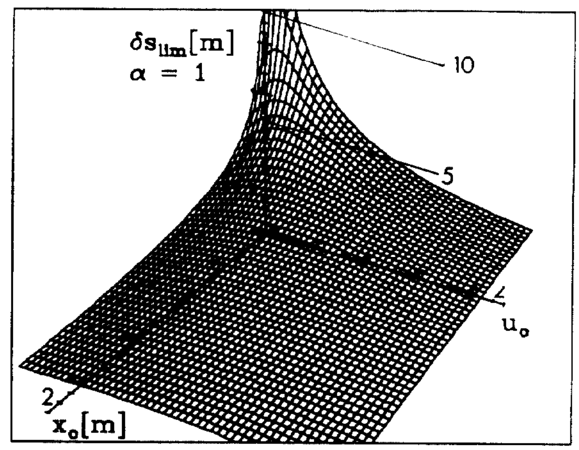

Fig. 3 : Maximum discretization for a 1D-sextupole as functions of x_0 and u_0.

In this example, the breakdown of the Taylor series at a certain limit of Δs can be understood easily by the fact that the solution becomes infinite for a real value of Δs. However, for complex poles, the breakdown of the Taylor series is not so obvious. Consider for example the pendulum motion :

$$\ddot{x} + \sin x = 0 \qquad (12)$$

This equation is discussed in detail in ref.[1] and we just give the results. The exact solution is :

$$x(t) = 2 \arcsin [k\, sn(K(m) + t|m)] \qquad (13)$$

where $m = k^2 = \sin^2(x_0/2)$, and K(m) is the complete elliptic integral of the first kind. Although x(t) stays bounded for all (real) values of the time t, the associated Taylor series diverges when Δt exceeds a limiting value. The explanation for this effect is that the Jacobian elliptic function $sn(u|m)$ has a pole on the imaginary axis :

$$u_p = i\, K(1-m) \qquad (14)$$

The associated limit for the discretization of a Taylor-mapping for the pendulum in the case of

$u_0 = 0$ is then given by :

$$\Delta t^2 = K^2(m) + K^2(1-m) \quad (15)$$

For $x_0 = \pi/2$ this yields $\Delta t = 0.42 \times 2\pi$. In order to illustrate this interesting behaviour, we plotted the terms of the series as a function of the order N for $\Delta t = 0.2 \times 2\pi$, $0.42 \times 2\pi$ and $0.8 \times 2\pi$ (Fig. 5). The first curve is well below the critical limit and hence a decrease of the terms with order can be seen which indicates convergence. The second case is just at the limit of convergence, while the third one obviously diverges.

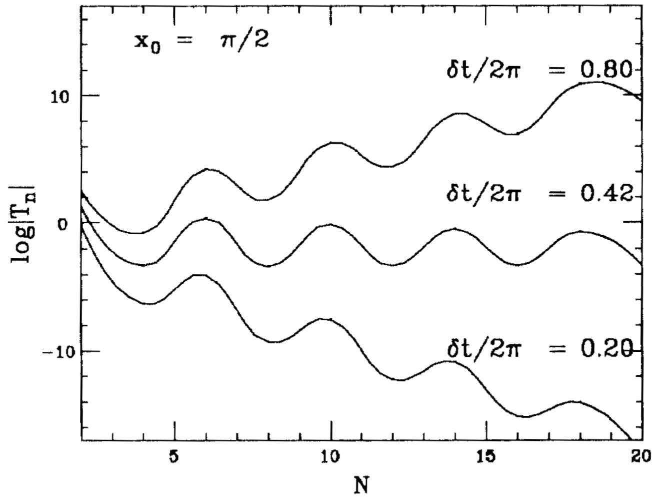

Fig. 4 : Taylor terms for the pendulum as functions of the order N for three discretizations $\Delta t/2\pi = 0.20, 0.42, 0.80$.

It should be noted that the phenomenon of complex poles arises also for the case of a pure octupole. This is described by the equation :

$$\ddot{x} + \gamma x^3 = 0 \quad (16)$$

with the solution for $u_0 = 0$ given by :

$$x = x_0 \, cn \left(\gamma^{1/2} x_0 s \,\middle|\, \tfrac{1}{2} \right) \quad (17)$$

For the sextupole (Eq. 7)) there exists only a real pole because the elliptic function occurs in the numerator as well as in the denominator, turning the complex poles into removable singularities.

4. The 2-D sextupole and tracking to computer accuracy.

Two dimensional motion in a sextupole is described by the nonlinear system of equations :

$$\ddot{x} = -\alpha (x^2 - y^2) \quad (18)$$

$$\ddot{y} = 2\alpha x y \quad (19)$$

The exact solutions are not known, and therefore also the location of the poles in the complex plane is unknown. Since the 1-D case is contained in Eqs. (18) and (19), an upper limit for the discretization Δs is given by Eq. (11) for $u_0 = y_0 = v_0 = 0$. We can apply the method of repeated differentiation to Eqs.(18) and (19) as before. Using this technique, we wrote the tracking-code (ACTP). Before tracking starts, a preprocessor determines the necessary order of the mapping (w.r.t. the sextupole length Δs) which solves Eqs. (18) and (19) to computer accuracy. This means that the Taylor term of one order higher must be smaller than the smallest number the computer can deal with in addition. It is necessary to input the range of initial conditions for which this accuracy should hold (e.g. the physical aperture). Fig. 5 shows an example of a number of Taylor terms created by a random choice of such initial conditions. As can be seen the necessary order in this case is N=12 for a 16 digit computer accuracy (IBM - double precision). If no converegence in this sense is found up to 20th order (the limit of this code), the sextupole will be split into two equal pieces by the program.

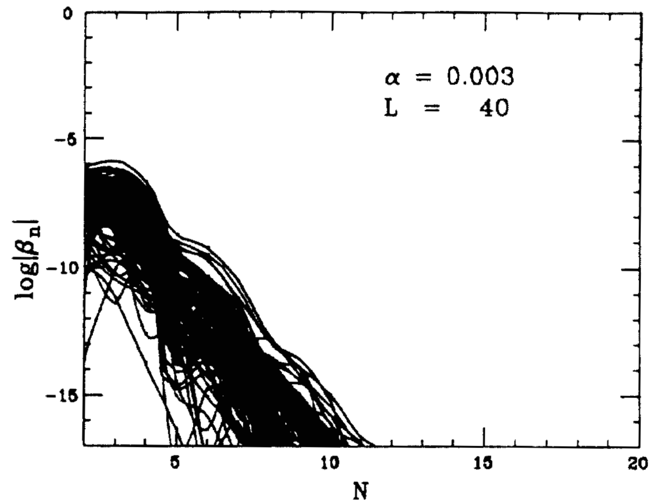

Fig. 5 : Taylor terms as functions of the mapping order for a random set of 100 initial conditions distributed across the entrance plane of a 2D-sextupole.

5. Conclusions

The comparison of Taylor-series mappings with the exact solution of the one-dimensional equations of motion for a charged particle in a sextupole permits the derivation of explicit expressions for the convergence. We found the existence of a limiting step-size, which depends in general on the initial conditions (x_0, u_0) but not on the order of the mapping.

These results are similar for all multi-polar elements, and restrict tracking of accelerator lattices by Taylor mappings to a maximum stepsize. Hence, concatenation of too many nonlinear elements will become divergent even for arbitrary high orders in the expansion of a map. This restriction of concatenation applies also to kick-codes, although the zero length of a single element guarantees that the mapping does not diverge element by element. In practice, the convergence of a concatenated map can be tested by comparing results for different step sizes.

6. Acknowledgements

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References

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