THIRD-ORDER COMBINED FUNCTION BENDING MAGNETS

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Summary

A complete third-order description of a charged particle optical system must include a combined function bending magnet. The magnetic field can have dipole, quadrupole, sextupole, and octupole terms. Midplane symmetry is assumed. Each additional order in the optical analysis requires inclusion of an additional multipole in the field expansion. First-, second-, and third-order expansions require quadrupole, sextupole, and octupole terms respectively. Third-order matrix elements may be derived by an iterative Green's function solution of the differential equations of motion. Third-order transfer matrix elements arise not only from third-order terms in the equations of motions, but also from the cascading effect of second-order terms. Solutions have been derived and have been incorporated into the computer programs TRANSPORT and TURTLE.

Introduction

Combined function bending magnets have been a standard component of accelerators and beam lines since the beginning of high-energy physics. Early synchrotrons such as the Cosmotron or the Bevatron used weak focusing where the normalized field index n took a value between 0 and 1, but distinctly different from .5. The invention of strong focusing led to combined function bending magnets where the absolute value of n was much greater than 1. The magnets then alternated in the sign of n, leading to more effective focusing and smaller apertures than was the case for accelerators with weak focusing. The AGS at Brookhaven and the PS at CERN were constructed according to this principle.

Still higher energy accelerators at Fermilab and CERN led to separated function design. The bending magnets had uniform central field and the focusing was done by a separate set of quadrupole magnets. These accelerators required such a large number of magnets that it was still a good approximation to consider both bending and focusing functions to be distributed uniformly about the circumference of the ring. In addition, the tune of the machine could now be controlled independently of the bending magnets.

Still the combined function bending magnet remained a useful tool for accelerator builders. The larger accelerators required smaller accelerators as booster devices. The Fermilab booster uses combined function bending magnets. The SLAC linear collider requires that the bending functions be as distributed as possible to limit energy losses due to synchrotron radiation. The focusing function must then be incorporated into the bending magnets leading to combined-function bending magnets.

The theory of charged-particle optics[1] was extended beyond the linear analysis in a paper by Brown, Belboach, and Bonin.[2] Additional analysis was performed in a SLAC summer study by Streib.[3] The set of second-order matrix elements for the transverse particle coordinates and their longitudinal derivatives was calculated by Brown.[4] Among other things, a second-order analysis allowed correction of the momentum dependent focusing of the beam by a curvature of the pole face of the bending magnets. The longitudinal matrix elements were later calculated by Brown, Servranckx, and Carey.[5]

In this paper we describe the calculation of the third-order transfer matrix elements of the central portion of a combined function bending magnet. The third-order representation of the fringing field will be described in a separate paper in this conference by Sagalovsky.[6] Let us begin with a description of the configuration of the magnetic field and the equations of motion.

Representation of the Magnetic Field

The assumption here is that the field configuration possesses midplane symmetry. By this we mean that the scalar potential from which the magnetic field is derived is an odd function of the vertical distance from the magnetic midplane. Transfer matrix elements have been calculated to second order for cases of violation of midplane symmetry and are described elsewhere.[7]

In the magnetic midplane a third-order expansion of the magnetic field is given by:

$$B_x = 0$$

$$B_y = B_o(1 - nx + \beta x^2 + \gamma x^3 + \ldots) \qquad (1)$$

$$B_s = 0$$

Because of Maxwell's equations, the midplane expansion of the magnetic field uniquely determines the field at all points off the magnetic midplane also. No additional coefficients are required for the full representation. The simplest method of determining the complete form is to require the magnetic scalar potential to satisfy Laplace's equation. When this is done, the complete form of the magnetic field is:

$$B_x = B_o\left[- nhy + 2\beta h^2xy + 3\gamma h^3x^2y - (\gamma + \tfrac{1}{3}\beta + \tfrac{1}{6}n)h^3y^3\right] \qquad (2)$$

$$B_y = B_o\left[1 - nhx + \beta h^2x^2 + \tfrac{1}{2}(n-2\beta)h^2y^2 + \gamma h^3x^3 - \tfrac{1}{2}(6\gamma+2\beta+n)h^3xy^2\right]$$

$$B_s = 0$$

Maxwell's equations continue to require the longitudinal component of the field to be zero since we are dealing with only the central portion of the field where there is no longitudinal dependence of either of the transverse components.

Equations of Motion

With no approximations or truncations by order, the equations of motion of a charged particle in a magnetic field are:

$$x'' - h(1+hx) - \frac{x'}{T'^2}\left[x'x'' + y'y'' + (1+hx)(hx'+h'x)\right]$$

$$= \frac{q}{p} T' \left[y'B_s - (1+hx)B_y\right] \qquad (3)$$

$$y'' - \frac{y'}{T'^2}\left[x'x'' + y'y'' + (1+hx)(hx'+h'x)\right]$$

$$= \frac{q}{p} T' \left[(1+hx)B_x - x'B_s\right]$$

All derivatives are with respect to distance s along the reference trajectory. The charge of the particle in question is q. The momentum p is given in terms of the reference momentum p_o and the fractional momentum deviation δ as $p = p_o(1+\delta)$. The quantity h is the curvature of the reference trajectory in the magnetic field. It is the reciprocal of the radius of curvature. In a uniform magnetic field h is constant and therefore h' is zero. The letter T represents the distance along a particular orbit. Its derivative T' then is the differential ratio of distance along a given orbit to that along the reference trajectory. The value of T' is given by

$$T'^2 = x'^2 + y'^2 + (1+hx)^2 \qquad (4)$$

From the above equations and the third-order expansion of the magnetic field, we can derive the complete set of equations of motion expanded to third order.

$$x'' + (1-n)h^2x = h\delta + (2n-1-\beta)h^3x^2 + (\beta-\tfrac{1}{2}n)h^3y^2$$

$$+ \tfrac{1}{2}h(x'^2 - y'^2) + (2-n)h^2x\delta - h\delta^2$$

$$+ (2\beta+\gamma-n)h^4x^3 + (3\gamma + 3\beta - \tfrac{1}{2}n)h^4xy^2$$

$$+ \tfrac{1}{2}(4-3n)h^2xx'^2 + \tfrac{1}{2}nh^2xy'^2$$

$$- nh^2x'yy' + (\beta-2n+1)h^3x^2\delta + \tfrac{1}{2}(n-2\beta)y^2\delta$$

$$+ \tfrac{3}{2}hx'^2\delta + \tfrac{1}{2}hy'^2\delta$$

$$+ (2-n)h^2x\delta^2 + h\delta^3 \qquad (5)$$

$$y'' + nh^2y = 2(\beta-n)h^3xy + hx'y' + nh^2y\delta$$

$$+ (3\gamma+4\beta-n)h^4x^2y - \tfrac{1}{6}(6\gamma+2\beta+n)h^4y^3$$

$$+ (n-2)h^2xx'y' - \tfrac{1}{2}nh^2x'^2y - \tfrac{3}{2}nh^2yy'^2$$

$$+ 2(n-\beta)h^3xy\delta + hx'y'\delta - nh^2y\delta^2$$

Solutions

The solution of the equations of motion are obtained by iteration. The first-order solutions for x, x', y, and y' are substituted into the right side of equations (5). The inhomogeneous non-linear equation is then solved by considering the right side to be a driving term, and expressing the solutions as Green's function integrals of the driving terms. The first iteration is complete to second order and also contains the single integrals of the third-order driving terms. A second iteration involves substituting this solution into the right sides of equations (5). The second-order solution is unchanged. The third-order solution contains additional terms which are the double Green's function integrals of products of the second-order driving terms.

Let the components x_i represent the six-vector x, x', y, y', ℓ, and δ. The previously undefined quantity ℓ represents the longitudinal separation between a given particle and one following the reference trajectory. We will not discuss this quantity any further, instead using the letter ℓ as an index over which summations can be made. The two differential equations (5) can now be schematically represented by the generic equation

$$x_i'' + k_i^2x_i = \sum_j D_{ij}x_j + \sum_{jk} E_{ijk}x_jx_k + \sum_{jk\ell} F_{ijk\ell}x_jx_kx_\ell \qquad (6)$$

The solution to third order may be represented as

$$x_i(1) = \sum_j R_{ij}x_j(0) + \sum_{jk} T_{ijk}x_j(0)x_k(0) + \sum_{jk\ell} U_{ijk\ell}x_j(0)x_k(0)x_\ell(0) \qquad (7)$$

The first-order matrix elements R, the second-order terms T, and the third-order terms U are all functions of position along the beam line. The single first-order term on the right side of equations (5) is the expression $h\delta$ which is the driving term for the first-order dispersion. The value of δ is not affected by iteration, so this term produces no higher-order effects. The second-order matrix elements T are given as single integrals of the driving terms E by:

$$T_{ijk} = \int_0^t G_i(t,\tau) \sum_{mn} E_{imn}R_{mj}(\tau)R_{nk}(\tau)d\tau \qquad (8)$$

The third-order elements are given as double integrals of products of second-order driving terms plus single integrals of third-order driving terms. Here we write the expression for the U matrix in terms of the R and T matrices. There are no explicit double integrals in the below equation since the T matrix is a single integral. The double integrals occur because the T matrix elements are themselves placed inside a Green's function integral.

$$U_{ijk\ell} = \int_0^t G_i(t,\tau) \sum_{mn} E_{imn} R_{mj}(\tau) T_{nk\ell}(\tau) d\tau \qquad (9)$$

$$+ \int_0^t G_i(t,\tau) \sum_{mn} E_{imn} T_{mjk}(\tau) R_{n\ell}(\tau) d\tau$$

$$+ \int_0^t G_i(t,\tau) \sum_{mnp} F_{imnp} R_{nj}(\tau) R_{mk}(\tau) R_{p\ell}(\tau) d\tau$$

Third-Order Matrix Elements

Space considerations do not permit the inclusion of the complete set of expressions for the third-order matrix elements for the transverse coordinates. There are 70 of them and the average expression has about twenty terms, most of which contain double integrals. It would also be next to impossible to transcribe the terms into this paper without error. A complete printing would be of questionable utility since the matrix elements are embodied in the computer programs TRANSPORT and TURTLE.[8,9] Anyone interested in evaluating the matrix elements would use one of these computer programs, rather than return to the original expression. As a sample, we include only one term. It is one of the most commonly encountered, being the second-order chromatic effect on the focusing of a system. The expression for U_{1266} follows:

$$U_{1266} = -2(2n-\beta) \int G_x(t,\tau) d_x(\tau) d\tau \qquad (10)$$

$$- (2-n) \int G_x(t,\tau) s_x(\tau) d\tau + 3h^2 \int G_x(t,\tau) c_x(\tau) s_x(\tau) d\tau$$

$$- (4-3n) h^3 \int G_x(t,\tau) c_x(\tau) s_x(\tau) d_x(\tau) d\tau$$

$$- 3(\gamma+2\beta-n) h^4 \int G_x(t,\tau) s_x(\tau) d_x^2(\tau) d\tau$$

$$- \frac{1}{2} (4-3n) h^4 \int G_x(t,\tau) s_x^3(\tau) d\tau$$

$$+ (2n-\beta) h^4 \int G_x(t,\tau) c_x(\tau) \int G_x'(\tau,\tau') d_x'^2(\tau') d\tau'$$

$$- (2n-\beta) h^4 \int G_x(t,\tau) s_x(\tau) \int G_x(\tau,\tau') d\tau'$$

$$+ (2n-\beta)(2-n) h^5 \int G_x(t,\tau) s_x(\tau) \int G_x(\tau,\tau') d_x(\tau') d\tau'$$

$$+ (2n-\beta)(2-n) h^5 \int G_x(t,\tau) d_x(\tau) \int G_x(\tau,\tau') s_x(\tau') d\tau'$$

$$+ (2n-\beta)(2-n) h^5 \int G_x(t,\tau) \int G_x(\tau,\tau') s_x(\tau') d_x(\tau') d\tau'$$

$$+ 2(2n-\beta) h^5 \int G_x(t,\tau) s_x(\tau) \int G_x'(\tau,\tau') s_x(\tau') d_x(\tau') d\tau'$$

$$+ 2(2n-\beta) h^5 \int G_x(t,\tau) d_x(\tau) \int G_x(\tau,\tau') c_x(\tau') s_x(\tau') d\tau'$$

$$+ (2n-\beta) h^6 \int G_x(t,\tau) s_x(\tau) \int G_x(\tau,\tau') s_x^2(\tau') d\tau'$$

$$+ 2(2n-\beta)^2 h^6 \int G_x(t,\tau) s_x(\tau) \int G_x(\tau,\tau') d_x^2(\tau') d\tau'$$

$$+ 4(2n-\beta)^2 h^6 \int G_x(t,\tau) d_x(\tau) \int G_x(\tau,\tau') s_x(\tau') d_x(\tau') d\tau'$$

$$- h^2 \int G_x(t,\tau) c_x(\tau) \int G_x'(\tau,\tau') d\tau'$$

$$+ (2-n) h^3 \int G_x(t,\tau) c_x(\tau) \int G_x'(\tau,\tau') d_x(\tau') d\tau'$$

$$+ (2-n) h^4 \int G_x(t,\tau) s_x(\tau) \int G_x'(\tau,\tau') s_x(\tau') d\tau'$$

$$+ (2-n) h^4 \int G_x(t,\tau) \int G_x(\tau,\tau') c_x(\tau') s_x(\tau') d\tau'$$

$$+ (2-n)^2 h^4 \int G_x(t,\tau) \int G_x(\tau,\tau') s_x(\tau') d\tau'$$

$$+ \frac{1}{2} h^4 \int G_x(t,\tau) c_x(\tau) \int G_x'(\tau,\tau') s_x^2(\tau') d\tau'$$

$$+ h^4 \int G_x(t,\tau) s_x(\tau) \int G_x'(\tau,\tau') c_x(\tau') s_x(\tau') d\tau'$$

The outer integrals are taken from 0 to t, the inner ones from 0 to τ.

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