ANALYTICAL EXPRESSIONS FOR THE SMEAR DUE TO NONLINEAR MULTIPOLES

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Abstract

An analytical derivation of the horizontal smear due to sextupoles and octupoles is presented. A generalized expression for the horizontal smear due to all multipoles is derived. A two degree of freedom calculation yields the smear due to sextupoles and octupoles. Experimental observations of the smear induced by special sextupoles have been made at the Fermilab Tevatron and our calculations agree very well with the data over a wide range of conditions. The smear due to random and systematic multipole errors in the dipoles, before and after the insertion of lumped correctors, is calculated for the SSC lattice. Finally the smear due to random and systematic multipole errors in the Tevatron dipoles is computed.

Introduction

For an ideally linear motion, a particle trajectory in the phase space at a certain location along the ring maps out a perfect ellipse which is an invariant. In the presence of nonlinearities, however, the trajectory fluctuates about the ellipse from turn to turn. The rms fractional value of this fluctuation is called the *single particle smear.*

In a collider ring the region around the axis of the magnets where the particle motion is sufficiently linear defines the linear aperture of the accelerator. Based on past accelerator experience [1], the linear aperture for the SSC has been defined [2] quantitatively as the region within which the smear is less than 6.4% and the on-momentum tuneshift with amplitude is less than .005. These criteria were subjected to experimental verification during the beam dynamics experiment E778 [9] performed in the Fermilab Tevatron. Furthermore single and multiparticle tracking calculations were used to predict the smear for various accelerator conditions. These predictions were compared to the experimental results. The agreement is very good. However, it is useful to derive an analytic expression for the smear. First, such a calculation can be compared to experimental and tracking results. Agreement among the three methods would enhance one's confidence in the understanding of the particle motion in the linear aperture region. Second, one could use this formula for the computation of the smear in a machine, without resorting to extensive tracking.

This paper presents analytical formulæ for the smear computation due to both field errors and correction multipole insertions. First order perturbation theory has been used to calculate the distortion of the beam shapes in the two transverse planes due to the nonlinearities, thus giving rise to the expressions for the smear. In the particular case of octupoles and sextupoles the smear is expressed conveniently in terms of Collins' distortion functions [4], the contribution from the two multipoles being separable. As we shall see, this is not the case if one includes higher multipoles. A number of applications of these formulæ are presented at the end. Analytic derivation has been performed by Forest [5] in the complicated Lie algebra notation. Our formulæ are simple.

Smear Due to Normal Sextupoles

First we perform the one degree of freedom analysis. Consider the situation of only sextupoles in the ring. For first order perturbation, the distortion of the horizontal particle amplitude $\mathcal{A}_x$ at phase advance ψ_x is given by [3,6]

$$\delta\mathcal{A}_x(\psi_x) = \mathcal{A}_x^2\{[A_1(\psi_x)\sin\varphi_x - B_1(\psi_x)\cos\varphi_x] + [A_3(\psi_x)\sin 3\varphi_x - B_3(\psi_x)\cos 3\varphi_x]\}\ , \quad (1)$$

where φ_x is the instantaneous betatron phase such that $x = \mathcal{A}_x\cos\varphi_x$, ψ_x is the phase advance and B_1, B_3, are the Collins' distortion functions:

$$\begin{aligned} B_1(\psi_x) &= \frac{1}{2\sin\pi\nu_x}\sum_k \frac{\mathcal{S}_k^{(2)}}{4}\cos(\psi'_{xk}-\psi_x-\pi\nu_x), \\ B_3(3\psi_x) &= \frac{1}{2\sin 3\pi\nu_x}\sum_k \frac{\mathcal{S}_k^{(2)}}{4}\cos 3(\psi'_{xk}-\psi_x-\pi\nu_x), \end{aligned} \quad (2)$$

and the A's are the derivatives of the B's with respect to their argument. Also $\mathcal{S}_k^{(2)}$ is the strength of the k-th sextupole defined in Eq. (7) below. The summations above are over each sextupole located at the 'modified' phase advance ψ'_{xk}, which is equal to the usual Floquet phase ψ_{xk} if $\psi_{xk}\geq\psi_x$, and to $\psi_{xk}+2\pi\nu_x$ if $\psi_{xk}<\psi_x$.

The single particle smear at ψ_x is defined as

$$S_X(\psi_x) = \left(\frac{\langle(\delta\mathcal{A}_x)^2\rangle}{\mathcal{A}_x^2}\right)^{1/2}\ , \quad (3)$$

where $\langle\ \rangle$ denotes the average over *many* turns, or, equivalently over the instantaneous betatron phase φ_x. From Eq. (1), we get immediately

$$S_X^2(\psi_x) = \frac{1}{2}\mathcal{A}_x^2\left\{A_3^2(\psi_x)+B_3^2(\psi_x)+A_1^2(\psi_x)+B_1^2(\psi_x)\right\}\ . \quad (4)$$

If we consider the distortion functions as vectors $R_1^{(2)}=(B_1,A_1)$ and $R_3^{(2)}=(B_3,A_3)$ then the smear can be expressed as

$$S_X^2(\psi_x) = \frac{1}{2}\mathcal{A}_x^2\left\{|R_1^{(2)}|^2+|R_3^{(2)}|^2\right\}_{\psi_x}\ . \quad (5)$$

From the definition of the distortion functions, Eq. (2), we get

$$|R_1^{(2)}(\psi_x)| = \frac{\left|\sum_k \mathcal{S}_k^{(2)}e^{i\psi'_{xk}}\right|}{8|\sin\pi\nu_x|},\qquad |R_3^{(2)}(\psi_x)| = \frac{\left|\sum_k \mathcal{S}_k^{(2)}e^{i3\psi'_{xk}}\right|}{8|\sin 3\pi\nu_x|}\ . \quad (6)$$

Further insight can be obtained from the following property of the distortion functions: the distortion functions at another point $\psi+\Delta\psi$ downstream are given by the vectors $R_1^{(2)}$ and $R_3^{(2)}$ rotated through angles $\Delta\psi$ and $3\Delta\psi$ respectively if there is no sextupole between the two points. In passing through a *thin* sextupole of length $L\to 0$ and strength

$$\mathcal{S}^{(2)} = \lim_{L\to 0}\left[\left(\frac{\beta_x^3}{\beta_0}\right)^{1/2}\frac{B_y''L}{2(B\rho)}\right]\ , \quad (7)$$

with horizontal betatron function β_x and particle's magnetic rigidity $(B\rho)$, the B_p's are continuous while the A_p's jump by an amount $\mathcal{S}^{(2)}/4$. Thus the smear will be a constant between two sextupoles but will have a jump when a sextupole is crossed. This is demonstrated in Fig. (1) which is obtained by plotting the smear as given by Eq. (5) as a function of the phase advance around the machine. Sixteen sextupoles clustered in two groups of eight located at phase advances of approximately

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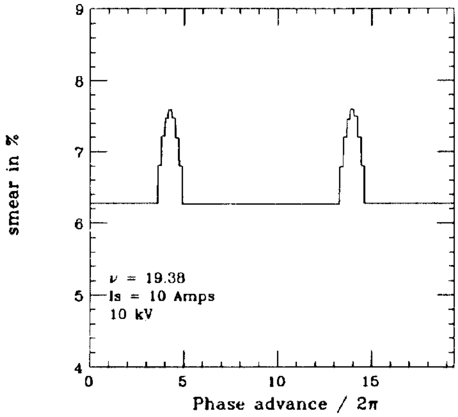

Figure 1: Smear versus phase advance, around the machine, as predicted from perturbation calculation.

$4.5 \times 2\pi$ and $14.5 \times 2\pi$ cause these jumps in the smear. In the special situation of having only one sextupole in the ring, the smear becomes a constant of motion.

Next we treat the two degree of freedom case. In two degrees of freedom the distortions of the horizontal and vertical amplitudes $\mathcal{A}_x$, $\mathcal{A}_y$ at phase advance ψ_x, to first order in the sextupole strength, are given by [3]

$$\begin{aligned}\delta\mathcal{A}_x &= \mathcal{A}_x^2[(A_1\sin\varphi_x - B_1\cos\varphi_x) + (A_3\sin 3\varphi_x - B_3\cos 3\varphi_x)]\\ &\quad -\mathcal{A}_y^2[2(\overline{A}\sin\varphi_x - \overline{B}\cos\varphi_x) + (A_s\sin\varphi_+ - B_s\cos\varphi_+)\\ &\quad -(A_d\sin\varphi_- - B_d\cos\varphi_-)],\end{aligned} \tag{8}$$

$$\delta\mathcal{A}_y = -2\mathcal{A}_x\mathcal{A}_y[(A_s\sin\varphi_+ - B_s\cos\varphi_+) + (A_d\sin\varphi_- - B_d\cos\varphi_-)]. \tag{9}$$

The distortion functions B_s, B_d, and $\overline{B}$, are given by

$$B_{s,d}(\psi_\pm) = \frac{1}{2\sin\pi\nu_\pm}\sum_k \frac{\overline{S}_k^{(2)}}{4}\cos(\psi'_{\pm k} - \psi_\pm - \pi\nu_\pm),$$

$$\overline{B}(\psi_x) = \frac{1}{2\sin\pi\nu_x}\sum_k \frac{\overline{S}_k^{(2)}}{4}\cos(\psi'_{xk} - \psi_x - \pi\nu_x), \tag{10}$$

and the A's are given by the derivatives of the B's. Here $\psi_\pm = 2\psi_y \pm \psi_x$ and $\nu_\pm = 2\nu_y \pm \nu_x$. The sextupole strength $\overline{S}^{(2)}$ is defined by

$$\overline{S}^{(2)} = \lim_{L\to 0}\left[\left(\frac{\beta_x\beta_y^2}{\beta_0}\right)^{1/2}\frac{B_y''L}{2(B\rho)}\right]. \tag{11}$$

In two degrees of freedom one can define three different kinds of smear:

$$S_{PP} = \left(\frac{\langle(\delta\mathcal{A}_p)^2\rangle}{\mathcal{A}_p^2}\right)^{1/2}, \qquad S_{XY} = \left(\frac{\langle(\delta\mathcal{A}_x\delta\mathcal{A}_y)\rangle}{\mathcal{A}_x\mathcal{A}_y}\right)^{1/2}, \tag{12}$$

where the subscript P stands for X and Y. Using Eqs (8) and (9) one can express the three smears in terms of the Collins' distortion functions as follows

$$\begin{aligned}S_{XX}^2 &= \frac{1}{2}\mathcal{A}_x^2(A_1^2 + B_1^2 + A_3^2 + B_3^2) + \frac{1}{2}\frac{\mathcal{A}_y^2}{\mathcal{A}_x^2}[(A_s^2 + B_s^2) + (A_d^2 + B_d^2)\\ &\quad + 4(\overline{A}^2 + \overline{B}^2)] - 2\mathcal{A}_y^2(A_1\overline{A} + B_1\overline{B}),\end{aligned} \tag{13}$$

$$S_{YY}^2 = 2\mathcal{A}_x^2(A_s^2 + B_s^2 + A_d^2 + B_d^2) \tag{14}$$

$$S_{XY}^2 = \mathcal{A}_y^2(A_s^2 + B_s^2 - A_d^2 - B_d^2). \tag{15}$$

For the explicit expressions of the smear in terms of the sextupole strengths and phases, see Ref. [7].

Smear Due to Normal Octupoles

The one degree of freedom calculation is performed first. The distortion of the horizontal amplitude $\mathcal{A}_x$ due to normal octupoles is given to first order in the octupole strength, by

$$\delta\mathcal{A}_x = \mathcal{A}_x^3[(A_1\sin 4\varphi_x - B_1\cos 4\varphi_x) + 2(A_2\sin 2\varphi_x - B_2\cos 2\varphi_x)] \tag{16}$$

where φ_x is the instantaneous betatron phase, and A_1, B_1, A_2, B_2 are the Collins' distortion functions. The B's are defined by

$$B_1(4\psi_x) = \frac{1}{2\sin 4\pi\nu_x}\sum_k \frac{\underline{S}_k^{(3)}}{8}\cos 4(\psi'_{xk} - \psi_x - \pi\nu_x),$$

$$B_2(2\psi_x) = \frac{1}{2\sin 2\pi\nu_x}\sum_k \frac{\underline{S}_k^{(3)}}{8}\cos 2(\psi'_{xk} - \psi_x - \pi\nu_x). \tag{17}$$

The octupole strength $\underline{S}^{(3)}$ is defined by

$$\underline{S}^{(3)} = \lim_{L\to 0}\left[\left(\frac{\beta_x^2}{\beta_0}\right)\frac{B_y'''L}{6(B\rho)}\right]. \tag{18}$$

Hence the horizontal smear given by Eq. (3) is

$$S_X^2 = \frac{1}{2}\mathcal{A}_x^4\left\{(A_1^2 + B_1^2) + 4(A_2^2 + B_2^2)\right\} \tag{19}$$

or

$$S_X^2 = \frac{1}{2}\mathcal{A}_x^4\left\{|R_1^{(3)}|^2 + 4|R_2^{(3)}|^2\right\} \tag{20}$$

where,

$$|R_1^{(3)}| = \frac{\left|\sum_k \underline{S}_k^{(3)}e^{i4\psi'_{xk}}\right|}{16|\sin 4\pi\nu_x|}, \quad |R_2^{(3)}| = \frac{\left|\sum_k \underline{S}_k^{(3)}e^{i2\psi'_{xk}}\right|}{16|\sin 2\pi\nu_x|}. \tag{21}$$

In two degrees of freedom the distortions of the horizontal and vertical amplitudes, $\mathcal{A}_x$ and $\mathcal{A}_y$ respectively, are given by

$$\begin{aligned}\delta\mathcal{A}_x &= \mathcal{A}_x^3[(A_1\sin 4\varphi_x - B_1\cos 4\varphi_x) + 2(A_2\sin 2\varphi_x - B_2\cos 2\varphi_x)]\\ &\quad -3\mathcal{A}_x\mathcal{A}_y^2[2(A_5\sin 2\varphi_x - B_5\cos 2\varphi_x)\\ &\quad +(A_3\sin 2\varphi_+ - B_3\cos 2\varphi_+) + (A_4\sin 2\varphi_- - B_4\cos 2\varphi_-)],\end{aligned} \tag{22}$$

$$\begin{aligned}\delta\mathcal{A}_y &= -3\mathcal{A}_x^2\mathcal{A}_y[2(A_6\sin 2\varphi_y - B_6\cos 2\varphi_y)\\ &\quad +(A_3\sin 2\varphi_+ - B_3\cos 2\varphi_+) - (A_4\sin 2\varphi_- - B_4\cos 2\varphi_-)]\\ &\quad +\mathcal{A}_y^3[2(A_8\sin 2\varphi_y - B_8\cos 2\varphi_y) + (A_7\sin 4\varphi_y - B_7\cos 4\varphi_y)].\end{aligned} \tag{23}$$

The distortion functions B_3, B_4, B_5, B_6, B_7, and B_8 are given by

$$B_{3,4}(2\psi_\pm) = \frac{1}{2\sin 2\pi\nu_\pm}\sum_k \frac{S_k^{(3)}}{8}\cos 2(\psi'_{\pm k} - \psi_\pm - \pi\nu_\pm),$$

$$B_{5,6}(2\psi_{x,y}) = \frac{1}{2\sin 2\pi\nu_{x,y}}\sum_k \frac{S_k^{(3)}}{8}\cos 2(\psi'_{(x,y)k} - \psi_{x,y} - \pi\nu_{x,y}),$$

$$B_7(4\psi_y) = \frac{1}{2\sin 4\pi\nu_y}\sum_k \frac{\overline{S}_k^{(3)}}{8}\cos 4(\psi'_{yk} - \psi_y - \pi\nu_y),$$

$$B_8(2\psi_y) = \frac{1}{2\sin 2\pi\nu_y}\sum_k \frac{\overline{S}_k^{(3)}}{8}\cos 2(\psi'_{yk} - \psi_y - \pi\nu_y), \tag{24}$$

Here $\nu_\pm = \nu_x \pm \nu_y$ and $\psi_\pm = \psi_x \pm \psi_y$. The octupole strengths $S^{(3)}$ and $\overline{S}^{(3)}$ are defined by

$$S^{(3)} = \lim_{L\to 0}\left[\left(\frac{\beta_x\beta_y}{\beta_0}\right)\frac{B_y'''L}{6(B\rho)}\right], \qquad \overline{S}^{(3)} = \lim_{L\to 0}\left[\left(\frac{\beta_y^2}{\beta_0}\right)\frac{B_y'''L}{6(B\rho)}\right]. \tag{25}$$

Then the three different smears given by (12) are

$$\begin{aligned}S_{XX}^2 &= \frac{1}{2}\mathcal{A}_x^4[(A_1^2 + B_1^2) + 4(A_2^2 + B_2^2)] + \frac{9}{2}\mathcal{A}_y^4[4(A_5^2 + B_5^2)\\ &\quad + (A_3^2 + B_3^2) + (A_4^2 + B_4^2)] - 12\mathcal{A}_x^2\mathcal{A}_y^2[A_2A_5 + B_2B_5],\end{aligned} \tag{26}$$

$$\begin{aligned}S_{YY}^2 &= \frac{9}{2}\mathcal{A}_x^4[4(A_6^2 + B_6^2) + (A_3^2 + B_3^2) + (A_4^2 + B_4^2)]\\ &\quad + \frac{1}{2}\mathcal{A}_y^4[4(A_8^2 + B_8^2) + (A_7^2 + B_7^2)] - 12\mathcal{A}_x^2\mathcal{A}_y^2[A_6A_8 + B_6B_8]\end{aligned} \tag{27}$$

and

$$S_{XY}^2 = \frac{9}{2}\mathcal{A}_x^2\mathcal{A}_y^2[(A_3^2 + B_3^2) - (A_4^2 + B_4^2)]. \tag{28}$$

Horizontal Smear Due to All Multipoles

In this section, we shall present a formula for the horizontal smear with the contributions from all higher multipoles without resorting to the use of distortion functions. The complete derivation can be found in Ref. [7].

The irrotational magnetic flux density can be written in general as

$$B_y + iB_x = B_0 \sum_{n=1}^{\infty} (b_n + ia_n)(x + iy)^n \,, \tag{29}$$

where b_n and a_n are the normal and skew multipole coefficients, respectively, of order $2(n+1)$. For example,

$$b_n = \frac{1}{n! B_0} \frac{\partial^n B_y}{\partial x^n} \,. \tag{30}$$

In the above, the vertical bending magnetic flux density B_0 as well as the field gradients of the focussing F and D quads have been excluded. Thus, Eq. (29) contains the contributions of all field errors as well as other inserted correction multipoles only. Since we are concerned with the isolated horizontal phase space only, Eq. (29) simplifies to $B_y = B_0 \sum_{n=1}^{\infty} b_n x^n$.

Then the smear S due to all higher multipoles, is given by

$$S^2 = \frac{1}{2} \sum_{p=1}^{\infty} \left| \sum_k \sum_{m=p}^{\infty} \frac{\mathcal{A}^{2m-2} S_k^{(2m-1)} f_p^{(2m-1)}}{\sin 2p\pi\nu} e^{\,i2p\psi_k'} \right|^2$$

$$+ \frac{1}{2} \sum_{p=0}^{\infty} \left| \sum_k \sum_{m=p}^{\infty}{}' \frac{\mathcal{A}^{2m-1} S_k^{(2m)} f_p^{(2m)}}{\sin(2p+1)\pi\nu} e^{\,i(2p+1)\psi_k'} \right|^2 . \tag{31}$$

Here $\mathcal{A}$ is the normalized amplitude, $\mathcal{A} = (2I\beta_0)^{1/2}$. Taking the thin lens approximation we define the strength of the k-th multipole, S_k, of length $L \to 0$ as

$$S_k^{(l)} = \lim_{L \to 0} \left[\frac{B_0 b_l}{(B\rho)} \left(\frac{\beta}{\beta_0} \right)^{(l+1)/2} \beta_0 L \right]_k \tag{32}$$

where $l = 2m - 1/2m$ for the $4m/4m+2$-th multipole. The coefficients $f_p^{(2m-1)}$ and $f_p^{(2m)}$ are defined by

$$f_p^{(2m-1)} = \frac{p}{2^{2m} m} \binom{2m}{m-p}, \quad f_p^{(2m)} = \frac{2p+1}{2^{2m+1}(2m+1)} \binom{2m+1}{m-p} \tag{33}$$

for the $4m$-th and $(4m+2)$-th multipole respectively.

Applications

The first application is on E778. Experiment E778 studied the nonlinear dynamics of transverse particle oscillations. Nonlinearities were introduced in the Tevatron by 16 special sextupoles. The smear was measured for different sextupole excitations (0 to 50 amperes), different tunes (19.38 to 19.42), and 3 kick amplitudes (5, 8 and 10 kV). Tracking calculations were done to simulate the experimental conditions and the smear was extracted from these calculations. We used Eq. (5) to compute the smear for the E778 Tevatron lattice for various conditions. The agreement between observation and prediction from perturbation theory is very good, as Fig. (2a) demonstrates. Also Fig. (2b) displays the comparison between perturbative calculations and tracking predictions. The agreement is also very good.

As a second application we shall calculate the smear in the Tevatron due to random and systematic errors in the dipoles. The Tevatron dipoles contain higher order multipole harmonics. The mean value of each multipole component is called the systematic error while the rms value constitutes the random error. We used Eq. (31) to calculate the smear and the errors are taken from Ref. [10]. For $B_0 = 4.4$ Tesla, at $\beta_0 = 100$ m with dipole length $L = 6.12$ m, at an amplitude of $\mathcal{A} = 5$ mm and tune of $\nu = 19.23$, the smear in the Tevatron due to random errors is $S = 1.04\%$. This result is in very good agreement with the measurements of the smear in the 'bare Tevatron' (nonlinearities turned off) performed as part of E778. For the same conditions, the smear in the Tevatron due to systematic errors, as calculated from Eq. (31), is $S = 0.80\%$.

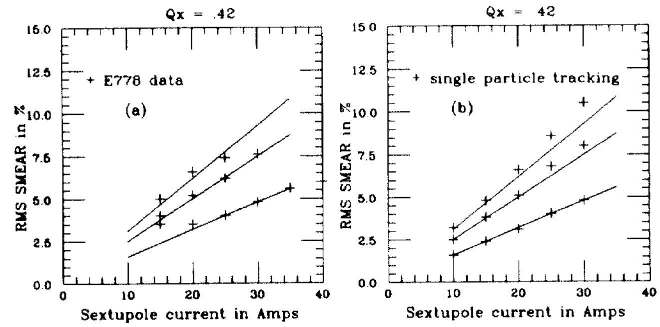

Figure 2: Smear vs sextupole excitation. Comparison between perturbative calculations (solid line) and (a) experimental data (E778), (b) tracking calculations. The 3 curves correspond to 2.25, 3.6 and 4.5 mm in amplitude.

	Smear (%)
No correction. b_2, b_3, b_4 random and systematic present.	7.09 ± 2.53
No correction. Random b_2, b_3, b_4 present.	7.07 ± 2.50
No correction. Systematic b_2, b_3, b_4 present.	0.39 ± 0.00
Correct random and systematic b_2, b_3, b_4. Random and systematic b_2, b_3, b_4 present.	0.43 ± 0.22
Correct random b_2, b_3, b_4. Random b_2, b_3, b_4 present.	0.43 ± 0.22
Correct random b_2. Random b_2, b_3, b_4 present.	0.71 ± 0.25
Correct systematic b_2, b_3, b_4. Systematic b_2, b_3, b_4 present.	0.0009 ± 0.00

Table 1: Summary of the results of the analytic computation of the smear in the SSC, with and without correction elements.

Finally we calculated the smear in the SSC due to random and systematic b_2, b_3 and b_4 given in Ref. [7]. We assumed an 'arcs only' SSC lattice with 320 cells and 12 dipoles per cell. The tune was 81.285 and the amplitude was 5 mm. Then we inserted correctors according to Neuffer's three lumped correction scheme [8] and recalculated the smear. The results are summarized in Table 1. The value of the smear fluctuates by large amount depending on the seed used. As a tolerance in design, one should allow the smear to vary by as much as say two deviations from the mean within the good field region.

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