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CALCULATING EMITTANCE FROM IMAGES OF UNDULATOR RADIATION[0]

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Abstract

A method is presented for calculating the emittance of an electron beam from scans of undulator radiation. Scans are affected by electron beam properties, radiation angular distribution, imaging pinhole diffraction, smearing from scanning, and differences between scan and beam axes, all of which are incorporated. Adding-in-quadrature is also discussed.

1. Introduction

The phase-space distribution of the photon beam from an undulator can be computed by convolving the angular distribution of the radiation from a single electron with the distribution of the electron beam. This distribution can be allowed to propagate forward in space, optionally filtered and diffracted by a pinhole, and scanned to create an image. For very low emittances, it is necessary to carry out these convolutions numerically. Only in the approximation that the intrinsic angular spread of the photons, the diffractive and smearing effects of the pinhole, and the smearing effects of the non-zero-aperture scanner can all be represented by the convolution of many gaussian functions can a purely analytic approach yield tractable expressions. This approximation will be exhibited in parallel with a more exacting approach, and the two methods compared.

The topic is covered more exhaustively in reference 1. An application is given for the low-emittance lattice of the storage ring PEP [2].

2. Electron and Photon Phase-Space Distributions

The phase-space distribution for electrons in either transverse plane is obtained by convolving the mono-energetic distribution with the energy distribution, giving

$$(2.1)\qquad \psi_e(q_e, q_e') = N_e\, e^{-(a_e q'^2 + 2b_e q q' + c_e q^2)/2}$$

where

$$a_e = \beta/\varepsilon - (\alpha\eta + \beta\eta')^2/d_e \quad ,$$

$$b_e = \alpha/\varepsilon - (\alpha^2\eta\eta' + \alpha\gamma\eta^2 + \gamma\beta\eta\eta' + \alpha\beta\eta'^2)/d_e \quad ,$$

$$c_e = \gamma/\varepsilon - (\gamma\eta + \alpha\eta')^2/d_e \quad ,$$

and

$$d_e = \varepsilon^2/\sigma^2_\delta + \varepsilon\cdot(\gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2) \quad ,$$

where σ_δ is the fractional energy spread, q_e is either x or y (assumed to be uncoupled), α, β, and γ are the Twiss parameters, and η is the dispersion.

The "intrinsic" radiation angular distribution, due to one electron passing through the undulator, is denoted by $\psi_{\gamma i}(x'_{\gamma i}, y'_{\gamma i})$, where $q'_{\gamma i} = q'_\gamma - q'_e$, with q standing for either x or y and q'_γ being the slope of the photon trajectory relative to the central orbit of the electrons.

I assume that $q_\gamma = q_e$. Strictly, this is not true, since the non-zero length of the undulator leads to an effectively non-zero intrinsic photon beam size (depth-of-field effect). The intrinsic photon beam spatial and angular distributions can be characterized by gaussian parameters σ_γ and σ'_γ given by [3]

$$(2.2)\qquad \sigma_\gamma = (2\lambda L)^{1/2}/(4\pi) \quad \text{and} \quad \sigma'_\gamma = (\lambda/2L)^{1/2} \quad .$$

where L is the undulator length and λ is the radiation wavelength, assumed to be a harmonic of the fundamental. Since even for the PEP low-emittance ($\varepsilon < 6.4$ π-nm-rad at 7.1Gev) lattice, σ_γ is much smaller than the electron beam size, it is ignored in what follows.

The phase-space distribution of the photon beam at the source is given by

$$(2.3)\qquad \psi_\gamma(x_\gamma, x'_\gamma, y_\gamma, y'_\gamma) = \int_{-\infty}^{\infty} dx'_{\gamma i}\, dy'_{\gamma i}\, \psi_{\gamma i}(x'_{\gamma i}, y'_{\gamma i})\, \psi_{xe}(x_e, x'_e)\, \psi_{ye}(y_e, y'_e)$$

with $q_e = q_\gamma - q_{\gamma i}$.

Far from an N-period undulator ($N \gg 1$) and for frequencies close to a harmonic of the fundamental, the intrinsic angular distribution is [4]

$$(2.4)\qquad \psi_{\gamma i}(\theta) = \frac{\sin^2(N\pi(\omega/\Omega - h) - kL\theta^2/4)}{(N\pi(\omega/\Omega - h) - kL\theta^2/4)^2} \quad ,$$

where ω is the frequency of the selected by the monochromator, Ω is the frequency of the first harmonic, h is the harmonic number, $\lambda = 2\pi/k$ the radiation wavelength, L the undulator length, and $\theta^2 \equiv x'^2 + y'^2$.

Under some circumstances (see section 8) the intrinsic radiation angular distribution is approximately gaussian in θ, with a sigma of $\sigma'^2_\gamma = \lambda/(2L)$. In this case, $\psi_{\gamma i}$ is approximately the product of two guassians in $x_{\gamma i}'$ and $y_{\gamma i}'$ with sigma σ'_γ, and (2.3) becomes:

$$(2.3')\qquad \psi_\gamma(x_\gamma, x'_\gamma, y_\gamma, y'_\gamma) \simeq \psi_{x\gamma}(x_\gamma, x'_\gamma)\, \psi_{y\gamma}(y_\gamma, y'_\gamma)$$

$$(2.4)\qquad \psi_{q\gamma}(q_\gamma, q'_\gamma) \simeq N_\gamma\, e^{-(a_\gamma q'^2_\gamma + 2b_\gamma q_\gamma q'_\gamma + c_\gamma q^2_\gamma)/2} \quad ,$$

where

$$a_\gamma = a_e/(\sigma^2_\gamma d_\gamma) \quad , \qquad b_\gamma = b_e/(\sigma^2_\gamma d_\gamma) \quad ,$$

$$c_\gamma = c^2_e - b_e/d_\gamma \; , \qquad d_\gamma = a_e + 1/\sigma^2_\gamma \quad .$$

Propagation of the photon beam a distance z from the source occurs according to $q_\gamma(z) = q_\gamma(0) + z\cdot q'_\gamma(0)$ and $q'_\gamma(z) = q'_\gamma(0)$. Thus

$$\psi_\gamma(x_\gamma, x'_\gamma, y_\gamma, y'_\gamma, z) = \psi_{\gamma 0}(x_\gamma - z\cdot x'_\gamma, x'_\gamma, y_\gamma - z\cdot y'_\gamma, y'_\gamma) \quad ,$$

where $\psi_{\gamma 0}$ is given by either (2.3) or (2.3').

3. Image Formation without a Pinhole

Assume that a photon beam propagates a distance T to an image plane. The image $I_a(x, y, T)$, called an "angular image", is given by

$$(3.1)\qquad I_a(x,y,T) = \int_{-\infty}^{\infty} dx'\, dy'\; \psi_{\gamma 0}(x - Tx', x', y - Ty', y')$$

The source of the name "angular image" can be seen by finding the sigma of the image along one axis. If one uses (2.8) in (3.1), one obtains for each plane

$$(3.2)\qquad \sigma^2_a = \frac{a_\gamma - 2b_\gamma T + c_\gamma T^2}{c_\gamma a_\gamma - b^2_\gamma}$$

As $T \to \infty$, the angular properties of the source dominate the image. This gives rise to the name "angular image" for a situation with no imaging pinhole. If the contributions from dispersion and the radiation opening angle are small, then the angular image depends primarily on ε/β. The so-called "spatial image" is obtained using a pinhole to image the radiation; under the same assumptions, $\varepsilon\beta$ is the primary determinant of the spatial image sigma, σ_s. Hence, the emittance is roughly given by $\sigma_s\sigma_a$ [5].

4. Pinhole Imaging in the Geometric Optics Limit

Assume that a pinhole of radius A at z=P is used to image the beam at some z=I>P. If diffraction can

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be ignored (see section 6), it is valid to think of the photon beam as an ensemble of non-interfering rays. The resultant "geometric spatial" image is

$$I_{gs}(x,y) = \int_{R(x,x',y,y')} dx'\,dy'\,\psi_{\gamma 0}(x-x'T,x',y-y'T,y')$$

where the region R is defined by

$$(x - x'I)^2 + (y - y'I)^2 < A^2 ,$$

and $T=I+P$.

Assume that the pinhole can be treated as a slit extending, say in y, in order to get an image along a line parallel to the x-axis (see section 8), and that (2.3′) applies. If the variation of $\psi_{x\gamma 0}$ with angle is negligible for variations in angle of order A/P, then a good approximation to the exact result is obtained by convolving the image of a point source due to a non-zero-aperture pinhole with the image of the actual source due to a point pinhole. If the photon beam is gaussian, one can employ adding-in-quadrature. Defining $\sigma_{sl}=(TA/P)/\sqrt{3}$, one finds that for $\sigma_{sl} \leq \sigma_{pp}$, adding in quadrature and fitting a gaussian to the actual image give sigmas that are within 5%, while for $\sigma_{sl} \leq \sigma_{pp}/\sqrt{3}$, the results agree to within 1%.

If the imaging is source-dominated and the pinhole can be treated as a slit, the image sigma is

$$\sigma_{pp}^2 = \frac{I^2}{a_\gamma - 2b_\gamma P + c_\gamma P^2}$$

In the limit that $P \to \infty$, this expression depends only on c_γ, hence the name "spatial image".

5. Pinhole Image Formation with Diffraction

For sufficiently long wavelengths (see section 6), diffraction effects become important. If the phase-space distribution produced by a single electron passing through the undulator has little variation across the pinhole, one can think in terms of an ensemble of nearly-spherical waves (one for each electron) impinging on the pinhole. For large source-to-pinhole distances, P, this is equivalent to an ensemble of point sources in the source plane, with the relative intensity of each source being $\psi(-x'P, x', -y'P, y')$, where ψ is the source distribution function.

If the size of the photon beam produced by a single electron at the pinhole is much larger than the pinhole itself, then one expects to be able to apply this method, since the assumption that the phase-space distribution varies but little over the pinhole is then satisfied. The criterion is approximately that $P^2\sigma_\gamma'^2 + \sigma_\gamma^2 >> A^2$ (cf. (2.2)).

In the Fresnel approximation, the image formed by the pinhole can be computed by convolving the image made by a point pinhole with the Fresnel diffraction pattern formed by an on-axis point source illuminating a non-zero-aperture pinhole. This convolution is an implementation of the amplitude weighting scheme discussed in the previous paragraph.

The image calculated via this scheme is

$$I_d(x,y) = \int_{-\infty}^{\infty} d\chi\,d\xi\; D(x-\chi,y-\xi)\cdot\psi(-\chi/M,\chi/I,-\xi/M,\xi/I) \tag{5.1}$$

where D is given by (6.7) and $M \equiv I/P$.

6. Fresnel Diffraction by a Circular Hole

In this section I give an expression for the diffraction pattern due to a spherical wave from a near-axis source impinging on a circular hole, as well as criteria for judging when one needs to consider diffraction. I first define the following functions, analagous to the Fresnel integrals

$$\left.\begin{matrix} C(\Delta t) \\ S(\Delta t) \end{matrix}\right\} \equiv \int_0^A a\,da \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (c_1 a^2)\; J_0(c_2 \Delta t\; a) ,$$

where $c_1 \equiv \pi/\lambda\cdot(1/I+1/P)$, $c_2 \equiv 2\pi/(\lambda I)$, and $\Delta t = (\Delta x , \Delta y)$ is the vector in the image plane from the diffraction pattern center to the image point.

The diffraction pattern is

$$D(\Delta x, \Delta y) = [C(\Delta t)]^2 + [S(\Delta t)]^2$$

This diffraction pattern does not depend on the source point **q**. Only the position **t** of the center of the pattern depends on **q**, through the point-pinhole-imaging relation $\mathbf{t}=-I\mathbf{q}/P$. Thus, (5.1) is valid if the Fresnel approximation is valid.

The Fresnel approximation is valid if

$$\frac{((2\sigma_u)^2 + A^2)^2}{P^3} << 8\lambda \quad \text{and} \quad \frac{((2\sigma_{ps})^2 + A^2)}{I^3} << 8\lambda$$

where $q=\pm 2\sigma_u$ characterizes the spatial extent of the photon beam at the undulator, and $t=\pm 2\sigma_{es}$ characterizes the spatial extent of the geometric image for a point pinhole (see equation (4.17)).

Diffraction may be ignored altogether (or else treated as a small effect to be added in quadrature) when

$$\frac{(2\sigma_u)^2 + A^2}{P} << 2\lambda \quad \text{and} \quad \frac{(2\sigma_{ps})^2 + A^2}{I} << 2\lambda$$

Note that when for both conditionals the terms on each side of the inequality are of the same order-of-magnitude, one obtains Fraunhofer-like diffraction patterns.

7. Effects of Image Scanning

Assume that images are scanned by a square aperture with sides of length 2h, which is scans in directions perpendicular to its sides, which are parallel to the axes in the u-v coordinate system, which may be tilted by an angle θ relative to the de-coupled x-y system for the electron beam.

A scan along the u axis, S(u), of an image I(x, y) is expressed analytically as

$$S(u) = \int_{u-h}^{u+h} d\hat{u} \int_{-h}^{h} d\hat{v}\; I(x(\hat{u}, \hat{v}), y(\hat{u}, \hat{v})) .$$

There are two effects combined in this equation which I will investigate separately, namely, the effect of non-zero h and the effect of non-zero θ.

Setting $\theta=0$, and assuming that I(x, y) is an uncoupled bi-gaussian in x and y, one obtains a convolution of a gaussian with a square aperture. Thus, if $\sigma_{ix} > h/\sqrt{3}$, the sigma of the scanned image is accurately given by adding the image sigma σ_{ix} in quadrature with $h/\sqrt{3}$.

Next, let $h \to 0$ and allow θ to be non-zero. In this case, $S(u) = I(u\cdot\cos\theta, -u\cdot\sin\theta)$. If I(x, y) is bi-gaussian, the scan is gaussian with sigma

$$\sigma_{sc}^2 = \sigma_{iy}^2\,\sigma_{ix}^2 \,/\, (\sigma_{iy}^2 \cos^2\theta + \sigma_{ix}^2 \sin^2\theta)$$

If $\sigma_{ix} \ll \sigma_{iy}$ or $\sigma_{ix} \gg \sigma_{iy}$, the sigma of the scanned image, σ_{sc}, depends strongly on θ, making the system potentially sensitive to tilts of the beam x-y distribution relative to the scanning axes. Such tilts can be due to poor alignment or to coupling of the horizontal and vertical betatron motions in the storage ring. In storage rings, one typically has a vertical emittance that is considerably smaller than the horizontal emittance, indicating that precautions need to be taken against erroneous conclusions from tilted scans, which might make the vertical emittance seem much larger than it actually is, without appreciably affecting the horizontal emittance. This effect may be mitigated by non-zero pinhole size, diffraction, and the non-zero size of the scanning aperture, all of which will (in a certain regime) broaden the smaller sigma much more than the larger.

8. Calculating Emittance from Scanned Images

The point of the above is to allow calculation of the emittance of the electron beam that generates the X-rays. Strictly, only one type of scan (angular or spatial) is needed for each plane; however, use of both types reduces errors, due, for example, to poor knowledge of the lattice functions.

For large emittances, abstracting the emittance from the scans is typically straight-forward, while for small emittances it may involve considerable numerical computation. The difference is that for a large emittance machine, the effects of non-zero pinhole size, diffraction, and the non-zero size of the scanning aperture can typically be treated by adding in quadrature (depending on the parameters of the imaging system), while for low emittance machines these effects are significant or even dominant, particularly for the vertical plane.

The "high-emittance" or "clean imaging" regime is defined by the following set of conditions, gleaned from the development above (most of the variables are defined above):

1. For spatial and angular scans, in both planes:
   a. The intrinsic radiation angular distribution must be accurately gaussian, or else small compared to the electron beam divergence.
   b. $h/\surd 3<\sigma_i$, where σ_i is the image sigma.
2. For spatial scans only, in both planes:
   a. $AT/(P\surd 3)< \sigma_{pp}$ (unless diffraction dominates).
   b. The angular divergence of the beam must be much larger than A/I (slit approximation).
   c. The beam size at the pinhole must be much larger than A (slit approximation).
   d. The beam size at the pinhole due to a single electron must be much large than A (diffraction approximation).
   e. The diffraction pattern must be accurately gaussian, or small compared to σ_{pp}.

Assuming that all these conditions hold, and assuming also for simplicity that there is no relative tilt of the scan and beam coordinate systems, the sigmas of the angular and spatial scans are calculated as, respectively,

$$\Sigma_s^2 = h^2/3 + + \sigma_{ph}^2 + \sigma_{pp}^2 ,$$

and

$$\Sigma_a^2 = h^2/3 + \sigma_a^2 ,$$

where σ_{ph} is either $AT/(P\surd 3)$ or some sigma characterizing the diffraction pattern.

Given the complexity of the dependence of the results on ε, it is not generally possible to solve for ε in terms of the angular and spatial sigmas; however, the problem can be dealt with numerically by minimizing the following function:

$$F(\varepsilon) = (\Sigma_a^* - \Sigma_a(\varepsilon))^2/\Delta\Sigma_a^{*2} + (\Sigma_s^* - \Sigma_s(\varepsilon))^2/\Delta\Sigma_s^{*2}$$

where starred quantities are measured values and the ΔΣ's are uncertainties in the measured sigmas. Note that the functions Σ_a and Σ_s also depend upon the lattice functions (i.e., β, α, η, and η'), and upon the imaging system parameters (i.e., A, P, I, and h). These parameters must must be measured independently and the uncertainty in their values must be folded into the uncertainty in the calculated value of the emittance. This is done by minimizing F for, say, β and β+δβ to obtain two values of ε, from which one computes ∂ε/∂β and hence the contribution to the error in ε due to uncertainty in one's knowledge of β, in this example.

When one or more of the conditions listed above are violated, it becomes necessary to do certain of the convolutions numerically. This is the situation in the "low-emittance" or "poor imaging" case. In this regime, the procedure for calculating ε is similar but requires more computation. The function F is again minimized, but Σ_a and Σ_s must be computed from fits to intensity profiles computed by numerically convolving the many effects described above, or by direct fits of computed images to the actual image scans.

9. Application to the Storage Ring PEP

Both exacting and approximate analysis was done for two sets of data taken on the PEP-1B beamline, with PEP at 7.1Gev[6]. The table below summarizes the results and compares them with theory (from lattice codes). The exacting method shows both better accuracy and sensitivity to changes. ε is the total emittance, and $\kappa=\varepsilon_y/\varepsilon_x$.

		Calculated from Expt.	
	Theory	Approximate	Exacting
Lattice:			
On-energy:			
ε (π-nm-rad)	6.4	4.5	5.3 ±0.8
κ	0	0.02	0.04±0.2
Off-energy:			
ε (π-nm-rad)	3.7	4.3	3.8 ±0.5
κ	0	0.04	0.015±0.008

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