# EFFECTS OF FREQUENCY SPREADS ON BEAM BREAKUP INSTABILITIES IN LINEAR ACCELERATORS

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## Abstract

Structure mode frequency spreads are shown to have a rather different influence on beam breakup growths than betatron frequency spreads. The present analytic and numerical studies show that a finite spread in the breakup mode frequency leads to an algebraic decay of the beam breakup instabilities even if the quality factor $Q \to \infty$. Effects of stagger tuning are examined.

## Text

Beam breakup (BBU) instabilities result from the coupling between the transverse motions of the electron beam and the deflecting modes of the accelerating structures. [1] Control of BBU growth relies mainly on the reduction of this coupling. [2] This can be achieved either by restricting the transverse motions of the beam through external focusing [1-8] (e.g., solenoidal, [3] quadrupole or higher order focusing, [2,7] ion channel, [5,6] etc.) or by proper modification of the deflecting modes (e.g., stagger tuning, [2,9] lowering of the quality factor Q of the deflecting modes, [10] etc.) Both methods have been suggested early on and adopted.

Although the lowering of Q is of immense practical importance [10] to control BBU, its effect on BBU growth is not a matter of curiosity, for it merely introduces the well-known damping factor $\exp(-\omega_0 t/2Q)$ to the amplitude of the disturbances. Here, $\omega_0$ is the breakup mode frequency of the structure. Lesser known and of more theoretical interest are other stabilization mechanisms such as detuning and various methods of focusing. To treat these various mechanisms on equal footing, without the interference of damping due to finite Q, we may set $Q = \infty$ and then compare the resulting growths [11,7] when the various effects are individually incorporated. In this paper, we shall use this strategy and show that a spread in the breakup mode frequencies would be far more effective to control BBU than a spread in the betatron frequencies.

The effects of frequency spreads are analyzed here in a somewhat different manner than was adopted in the literature. [8,9,12,13] We begin with the BBU dispersion relationship [6,11]

$$(\omega - kv)^2 - \omega_c^2 = \varepsilon \omega_0^3/(\omega - \omega_0). \quad (1)$$

This dispersion relationship describes the excitation of the beam modes ($\omega - kv \simeq \pm \omega_c$) by the breakup modes ($\omega \simeq \omega_0$) on a continuous, coasting beam whose transverse displacement varies as $\exp(i\omega t - ikz)$. Here $\omega_c$ is the betatron frequency of the (linear) focusing field, v is the electron speed, and $\varepsilon$ is the dimensionless coupling constant [11] which is proportional to the beam current and transverse shunt impedance. The evolution of BBU is described by the Green's function

$$G(z,t) \sim \int d\omega \, \exp[i\omega t - ik(\omega)z], \quad (2)$$

where

$$k(\omega) = \frac{\omega}{v} \pm \frac{1}{v} \left(\omega_c^2 + \frac{\varepsilon \omega_0^3}{\omega - \omega_0}\right)^{1/2} \quad (3)$$

is obtained from the dispersion relation (1). A casual examination of Eq. (2) suggests that, at fixed z, the asymptotic behavior ($t \to \infty$) of $G(z,t)$ is dictated by the singularity $\omega \simeq \omega_0$ in Eq. (3). In the immediate neighborhood of this singularity, the term $\omega_c^2$, which represents the focal strength, is unimportant. A finite spread in the betatron frequency does not change the singular behavior of $k(\omega)$ either, and therefore cannot influence the asymptotic BBU growth. [11,7] In physical terms, the BBU is locked onto the breakup mode frequency ($\omega \simeq \omega_0$) so that neither a linear focusing field nor phase mixing due to a spread in the betatron frequency can break this locking in a long pulse beam if $Q = \infty$. [14]

On the other hand, a spread in the breakup mode frequencies might be expected to modify the BBU growth since a modification in $\omega_0$ alters the singular behavior of $k(\omega)$ in Eq. (3) altogether, and the asymptotic BBU growth would be modified accordingly. It was this observation which motivated the present study. Physically, a finite linewidth (due, for instance, to mechanical tolerance in the cavities) would contribute to a spread in the breakup mode frequency [9] and stagger tuning would introduce [2,9] a z-dependence in $\omega_0$; both might lead to a change in the structure of $k(\omega)$.

To examine these possibilities, we set $\omega_c = 0$ henceforth. The effects of a finite spread in the breakup mode frequency $\omega_0$ may be studied by simply replacing the factor $\varepsilon \omega_0^3/(\omega - \omega_0)$ in Eq. (1) by

$$\int d\omega_0 \, f(\omega_0) \varepsilon \omega_0^3/(\omega - \omega_0),$$

where $f(\omega_0)$ is the spectral distribution function normalized so that

$$\int d\omega_0 \, f(\omega_0) = 1.$$

We next calculate $k(\omega)$, insert it into Eq. (2), and determine the asymptotic behavior of $G(z,t)$. For the simple distribution function

$$f(\omega_0) = \begin{cases} 1/\Delta & ; \; |(\omega_0 - \bar{\omega}_0)/\bar{\omega}_0| < \Delta/2 \\ 0 & ; \text{ otherwise,} \end{cases} \quad (4)$$

a saddle point calculation yields, to two orders,

$$|G(z,t)| \simeq \frac{C}{\sqrt{Z}} A(\tau_1) \exp\left\{1.64W^{1/3} S(\tau_1)\right\}, \quad (5)$$

where $A(\tau_1)$ and $S(\tau_1)$ are shown in Fig. 1. In Eqs. (4) and (5), $\Delta$ represents the full width spread in the breakup mode frequency whose mean is $\bar{\omega}_0$, C is a normalization constant, $Z \equiv \bar{\omega}_0 z/v$, $T \equiv \bar{\omega}_0 t$, $W \equiv \varepsilon(T - Z)Z^2$,

$$\tau_1 \equiv (T - Z)\Delta^{3/2}/2Z\varepsilon^{1/2}, \quad (6)$$

$$S(\tau_1) = - \text{Im}\left[0.485 \, \bar{\omega}\tau_1^{2/3} \left[1 + 1/\bar{\omega}(\bar{\omega}^2 - 1)\tau_1^2\right]\right], \quad (7)$$
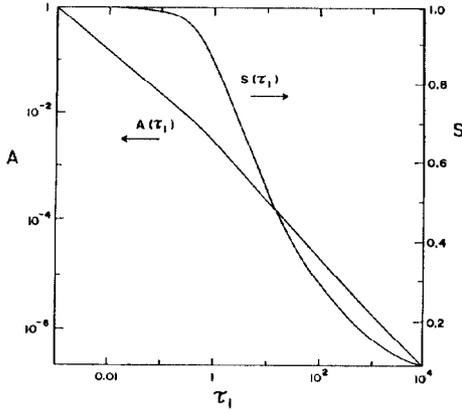
Fig. 1    The analytic solutions $S(\tau_1)$ and $A(\tau_1)$.

$$A(\tau_1) = \left| (\bar{\omega}^2 - 1)/\tau_1 \left[ 2\bar{\omega} - \tau_1^2(\bar{\omega}^2 - 1)^2 \right] \right|^{1/2}, \qquad (8)$$

and $\bar{\omega} = \bar{\omega}(\tau_1)$ is the (meaningful) solution to the transcendental equation

$$\tau_1^2(\bar{\omega}^2 - 1)^2 \ell n \left( \frac{\bar{\omega} - 1}{\bar{\omega} + 1} \right) - 1 = 0. \qquad (9)$$

The following asymptotic dependences of A and S might be useful: $S(\tau_1) \to 1$ for $\tau_1 \ll 1$ and $S(\tau_1) \to (\ell n \tau_1)^{1/2}/\tau_1^{1/3}$ for $\tau_1 \gg 1$. Over the range shown in Fig. 1, a reasonable extrapolation gives $A(\tau_1) \simeq 0.00328/\tau_1^{0.828}$ for $\tau_1 \le 0.4$ and $A(\tau_1) \simeq 0.00265/\tau_1^{1.062}$ for $\tau_1 \ge 0.4$. As will be shown in Fig. 2, this analytic solution (5) is an excellent approximation to the numerically integrated solution [except at T = Z, where the asymptotic solution (5) diverges].

The solution (5) and Fig. 1 reveal the following. First, for a given spread $\Delta$ in the breakup mode frequency, this spread has little effect on the BBU evolution if the pulse length (T), machine length (Z), beam current and shunt impedance ($\varepsilon$) combine in such a way that $\tau_1 < O(1)$, in which case the BBU evolves according to the classic solution of Panofsky and Bander. [1]  However, if $\tau_1 \gg O(1)$, Fig. 1 and Eq. (5) show that $|G(Z,T)|$ tends to zero like $1/T$      as $T \to \infty$. This algebraic decay is a result of phase mixing due to the finite spread in the breakup mode frequency and it begins to appear for $\tau_1 \ge O(1)$. This decay is weaker than the exponential decay associated with a finite Q of the cavity, but is much stronger than that produced by a finite spread in the betatron frequency.

The effects of finite linewidth $\Delta$ may also be studied by a direct numerical integration of the governing equations. We pretend that the normalized transverse displacement X(Z,T) of the beam is to be excited by N deflecting modes, the i-th mode has a frequency $\omega_{oi}$, given by $\omega_{oi}/\omega_o \equiv \bar{\omega}_i = (1 - \Delta/2) + (i - 1)\Delta/(N - 1)$, $i = 1, \ldots .N$. In keeping with the above model [cf. Eq. (4)], we assume that there is an equal probability (1/N) for each of these deflecting modes to be excited. Thus, the evolution X(Z,T) is governed by

$$\left( \frac{\partial}{\partial T} + \frac{\partial}{\partial Z} \right)^2 X = \frac{1}{N} \left( A_1 + .. + A_N \right), \qquad (10)$$
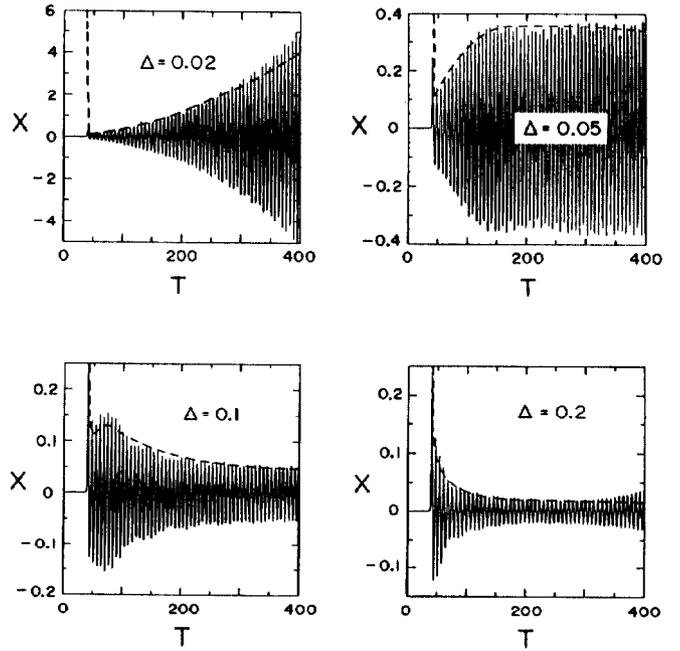


Fig. 2    Numerical solutions at various breakup mode frequency spreads ($\Delta$). The dotted curves represent the analytic solution.

where $A_i$, the normalized amplitude of the i-th mode, is excited by the transverse displacement X(Z,T) according to

$$\frac{\partial^2 A_i}{\partial T^2} + \bar{\omega}_i^2 A_i = 2\varepsilon X, \qquad i = 1, \ldots N. \qquad (11)$$

We assume initial rest condition and homogeneous boundary condition for these N + 1 equations (10), (11).  Only one non-trivial boundary condition is imposed at Z = 0: X(0,T) = 1 for 0 < T < 0.2 and is zero otherwise.  This form of the excitation mimics an impulse excitation and the numerical solution X(Z,T) may then be compared with the Green's function Eq. (5).  [The latter solution corresponds to the limit $N \to \infty$.]

Shown in Fig. 2 is the numerically integrated solution X(Z,T) according to (10), (11), with N = 16, for $\Delta$ = 0.02, 0.05, 0.1, 0.2.  We set Z = 40 and $\varepsilon$ = 0.000413 in this figure.  The dotted curves in this figure represent the analytic solution (5), in which the constant C is determined by a "one-point-fit" with the numerical solution.  The high values of $\Delta$ used here were chosen to accentuate their effects for T < 400.

Figure 2 clearly shows the stabilizing effect due to a finite spread in the breakup mode frequency. When the spread ($\Delta$) is small, as in the $\Delta$ = 0.02 case, its effect on BBU growth is negligible for T < 400, and the Green's function is basically that of Panofsky and Bander. [1]  As $\Delta$ is increased to 0.05, saturation of BBU growth begins to appear at T $\simeq$ 150, corresponding to $\tau_1$ = O(1).  For still higher values of $\Delta$, algebraic decay of the amplitude is observed. This result is somewhat different from Ref. 9.  It is interesting to note that the analytic solution (5), which contains two approximations to (10) and (11) [cf., $N \to \infty$ limit, and then asymptotic analysis], could accurately model the numerical solutions to (10), (11) with N = 16.  On the other hand, it is remarkable that the use of only sixteen modes in the numerical calculation could adequately model phase-mixing and accurately display the transition of the solutions when $\tau_1 \simeq O(1)$.  [The mild increase in the numerical solution X(Z,T) as T $\to$ 400 in the $\Delta$ = 0.2

1194

case in Fig. 2 results from the constructive interference of the finite number of modes used in the numerical integration].

Note that the transition time $\tau_1 = O(1)$ is the time when the phase mixing factor $\Delta\omega_o t$ is roughly balanced by the BBU exponentiation factor $1.64W^{1/3}$ [cf. Eq. (5)]. This result is perhaps not too unexpected from the structure of the dispersion relation (1). A full calculation, however, was required to identify the algebraic decay associated with a finite $\Delta$.
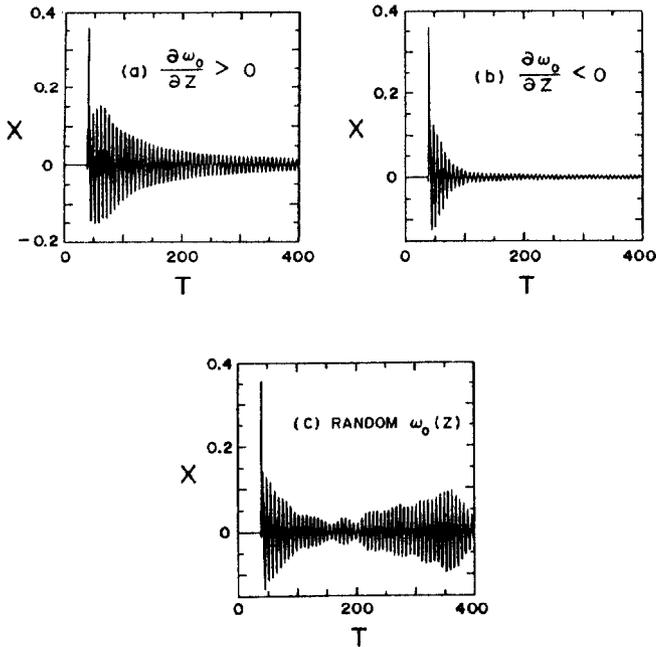


Fig. 3   Evolution of BBU when the breakup mode frequency $\omega_o$ is a function of Z over $0 \leq Z \leq 40$: (a) $\omega_o$ increases linearly by twenty percent. (b) $\omega_o$ decreases linearly by twenty percent. (c) $\omega_o$ is a random function of Z bounded between $0.9\ \bar{\omega}_o$ and $1.1\ \bar{\omega}_o$.

In summary, finite spreads in the breakup mode frequencies produce a considerably different effect on BBU from finite spreads in the betatron frequencies. Such a difference was transparent in the present mode coupling analysis. A spread in the breakup mode frequencies leads to an algebraic decay of BBU even if $Q \rightarrow \infty$. Asymptotic formulas and time scale for this phase mixing to occur are presented. They are in excellent agreement with numerical integration. Effects of stagger tuning was also examined numerically.

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14. If nonlinear effects are included, the locking phenomena mentioned here may be broken, as demonstrated in Ref. 7, where we showed that BBU can be stabilized by a sufficiently strong nonlinear focusing field even in the limit $Q \rightarrow \infty$.

The effect of stagger tuning may also be investigated by integrating Eqs. (10) and (11), where we now set $N = 1$ and assign a z-dependence to $\bar{\omega}_1 = \omega_o/\bar{\omega}_o$. Shown in Fig. 3 are the numerical solutions at $Z = 40$ for three cases, each of which contains a maximum variation of $\bar{\omega}_1$ of twenty percent over $0 \leq Z \leq 40$: (a) $\bar{\omega}_1 = 0.9 + 0.2\ (Z/40)$, (b) $\bar{\omega}_1 = 1.1 - 0.2\ (Z/40)$ and (c) $\bar{\omega}_1(Z)$ is a random function bounded by twenty percent variation about unity over $0 \leq Z \leq 40$.

Comparing Fig. 3 with the $\Delta = 0.2$ case of Fig. 2, we see that a non-constant $\omega_o(z)$ has a similar stabilizing influence as an intrinsic spread in the breakup mode frequency for the parameters under consideration. At the moment, we have not obtained analytic formulas similar to Eq. (5) for general $\omega_o(z)$, and no readily usable scalings emerged from the large body of numerical results which we obtained for general $\omega_o(z)$.