HIGH FREQUENCY DEPENDENCE OF THE COUPLING IMPEDANCE FOR A LARGE NUMBER OF OBSTACLES[1]

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Introduction

We have recently derived[2,3] an integral equation for the axial electric field at the pipe radius in the presence of an azimuthally symmetric cavity of arbitrary shape in a beam pipe of circular cross section. We have further shown that the local average of the coupling impedance over frequency decreases as $k^{-1/2}$ for high frequency, essentially independent of the cavity shape. In another paper[4], we extend the derivation to several cavities and obtain the high frequency behavior for a periodic cavity. In this case the real part of the impedance per cell is shown to vary as $k^{-3/2}$, in agreement with Heifets and Kheifets[5], and the imaginary part varies as k^{-1}, as required by causality.

In the present paper we analyze the case of N cavities and explore the high frequency behavior for large N, in an effort to understand the transition to a periodic structure. Not unexpectedly, the result depends critically on which of the limits ($k \to \infty$ or $N \to \infty$) is taken first.

Analysis

The starting point for the analysis is the integral equation obtained for the axial electric field in a single obstacle at the beam pipe radius.[2,3] Specifically we have

$$\int_0^g dz'\, G(z') \left[\hat{K}_p(z - z') + \hat{K}_c(z',z)\right] = j \qquad (2.1)$$

and

$$\frac{Z(k)}{Z_0} = \frac{1}{ka^2} \int_0^g dz\, G(z) \quad . \qquad (2.2)$$

Here $kc/2\pi$ is the frequency, a is the pipe radius, $Z_0 = 120\pi$ ohms is the impedance of free space, and the azimuthally symmetric obstacle, of general shape in the r,z plane, extends axially from $z = 0$ to $z = g$ at the pipe radius $r = a$. Apart from a constant and the factor $\exp(jkz)$, G(z) is the axial electric field for $r = a$ and $0 < z < g$.

The modified "pipe" kernel, $\hat{K}_p(u)$, has the form[2]

$$\hat{K}_p(u) = \frac{2\pi j}{a} \sum_{s=1}^{\infty} \frac{e^{jku - j b_s |u|/a}}{b_s}$$

$$\cong \frac{2\pi j}{ka^2} \begin{Bmatrix} 0 & , & u < 0 \\ \sum_{s=1}^{\infty} e^{juj_s^2/2ka^2} & , & u > 0 \end{Bmatrix} \quad , \qquad (2.3)$$

where $u = z - z'$, $b_s^2 = k^2a^2 - j_s^2$, and where the last form in Eq. (2.3) is obtained by averaging over frequency, with the dominant contributuion coming from $1 \ll j_s \ll ka$. For $|u| \ll ka^2$, the sum over s can be converted to an integral, leading to

$$\hat{K}_p(u) \cong \begin{Bmatrix} 0 & , & u < 0 \\ \dfrac{(j-1)\sqrt{\pi}}{a\sqrt{ku}} & , & u > 0 \end{Bmatrix} \quad . \qquad (2.4)$$

A similar analysis[2] for the "smoothed" high frequency limit of $\hat{K}_c(z',z)$ also leads to the same result, namely

$$K_c(z',z) \cong \begin{Bmatrix} 0 & , & z' > z \\ \dfrac{(j-1)\sqrt{\pi}}{a\sqrt{k(z-z')}} & , & z' < z \end{Bmatrix} \quad . \qquad (2.5)$$

The solution of Eq. (2.1) with the kernels in Eqs. (2.4) and (2.5) then yields the "smoothed" high frequency limit for the impedance for a single obstacle:

$$\frac{Z_0}{Z(k)} = Z_0 Y(k) = F_0(k),\ F_0(k) = \frac{(1+j)\,\pi a\sqrt{\pi k}}{\sqrt{g}} \quad . \qquad (2.6)$$

For several obstacles, it is easy to see that Eq. (2.1) can be generalized to

$$\sum_m \int_m dz'_m\, G(z'_m) \left[\hat{K}_p(z_n - z'_m) + \delta_{mn}\hat{K}_c(z'_m, z_n)\right] = j \ , \qquad (2.7)$$

where z'_m and z_n denote the variables z' and z within cavities m and n, and $\int_m dz'_m$ is over cavity m. The coupling between different cavities occurs through the pipe kernels, whereas the cavity kernels are diagonal. If we now use the high frequency kernels in Eqs. (2.4) and (2.5) for the diagonal terms and Eq. (2.3) for the pipe kernel in the coupling terms, it is clear that the only surviving contributions to the sum over m will be those for $z'_m < z_n$, that is $m \le n$. Specifically we obtain

$$\frac{2(1+j)\sqrt{\pi}}{a\sqrt{k}} \int_0^t \frac{dt'\, G_n(t')}{\sqrt{t - t'}} +$$

$$+ \frac{2\pi}{ka^2} \sum_{s=1}^{\infty} \sum_{m=1}^{n-1} \exp\left(\frac{j(n-m)Lj_s^2}{2ka^2}\right) \int_0^g dt'\, G_m(t') = 1 \ , \qquad (2.8)$$

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where $z'_m = mL + t'$, $z_n = nL + t$, and where we assume that we hav N identical cavities whose centers are spaced a distance L apart. We have also approximated $z_n - z'_m$ by $(n - m)L$ in the non-diagonal terms, corresponding to the assumption NL >> g. The impedance is then

$$\frac{Z(k)}{Z_o} = \frac{1}{ka^2} \sum_{m=1}^{N} \int_o^g dt\, G_m(t) \quad . \qquad (2.9)$$

Equation (2.8) can be simplified by writing

$$G_n(t') = \frac{(1 - j)\, a \sqrt{k}}{4\pi \sqrt{\pi} \sqrt{t'}}\, y_n \quad , \qquad (2.10)$$

leading to

$$y_n + \frac{(1 - j)\sqrt{g}}{a\sqrt{\pi k}} \sum_{s=1}^{\infty} \sum_{m=1}^{n-1} y_m \exp\left(\frac{j(n - m)Lj_s^2}{2ka^2}\right) = 1 \qquad (2.11)$$

and

$$\frac{Z(k)}{Z_o} = \frac{(1 - j)\sqrt{g}}{2\pi a\sqrt{\pi k}} \sum_{n=1}^{N} y_n \quad . \qquad (2.12)$$

Our task is to solve Eq. (2.11) for y_n and then use Eq. (2.12) to obtain the impedance. This can be facilitated by constructing the transform $w(h) = \sum_{n=1}^{\infty} h^n y_n$, in which case use of the convolution theorem leads to the solution

$$w(h) = \frac{h}{1 - h}\left[1 + \frac{(1 - j)\sqrt{g}}{2\pi a\sqrt{\pi k}}\, p(h)\right]^{-1} \quad , \qquad (2.13)$$

where

$$p(h) = \sum_{s=1}^{\infty} \sum_{\ell=1}^{\infty} h^{\ell} e^{j\frac{\ell L j_s^2}{2ka^2}} \cong \sum_{s=1}^{\infty} \frac{h}{1 - h - j\dfrac{L j_s^2}{2ka^2}} \quad . \qquad (2.14)$$

The last form of Eq. (2.14) holds in the range $ka^2 >> Lj_s^2$.

A simple approximation to Z(k) in Eq. (2.12) for large N can be obtained by evaluating

$$w[\exp(-1/N)] = \sum_{n=1}^{\infty} y_n e^{-n/N} \quad , \qquad (2.15)$$

where the exponential cut-off simulates the sum from n = 1 to N in Eq. (2.12). For $h \cong 1 - 1/N$, we find

$$w\left(1 - \frac{1}{N}\right) \cong N\left[1 + \frac{(1 - j)\sqrt{g}}{\sqrt{\pi k a^2}} \sum_{s=1}^{\infty} \frac{1}{\dfrac{1}{N} - j\dfrac{L j_s^2}{2ka^2}}\right]^{-1} . \qquad (2.16)$$

Let us first consider the limit $N \to \infty$. In this case we can use $\sum_{s=1}^{\infty} j_s^{-2} = 1/4$ to evaluate the sum over s, to obtain

$$N\, Z_o Y(k) \cong \frac{(1 + j)\, \pi a \sqrt{\pi k}}{\sqrt{g}} + \frac{j\pi k a^2}{L} \text{ , large N ,} \qquad (2.17)$$

the result obtained earlier[4] for a periodic structure. If instead, we assume that $1 << N << ka^2/L$, the sum over s can be converted to an integral over j_s from 0 to ∞ to give

$$N\, Z_o\, Y(k) \cong \frac{(1 + j)\, \pi a \sqrt{\pi k}}{\sqrt{g}}\left[1 + \frac{\sqrt{gN}}{\sqrt{\pi L}}\right] \quad . \qquad (2.18)$$

This limit corresponds to converting the sum over s to an integral in Eq. (2.11), leading to

$$y_n + \frac{1}{\pi}\frac{\sqrt{g}}{\sqrt{L}} \sum_{m=1}^{n-1} \frac{y_m}{\sqrt{n - m}} = 1 \quad . \qquad (2.19)$$

For large n, it is easy to show from Eq. (2.19) that the asymptotic form of y_n is

$$y_n \to \frac{\sqrt{L}}{\sqrt{gn}} \quad , \qquad (2.20)$$

leading to

$$N\, Z_o\, Y(k) \cong \frac{(1 + j)\, \pi a \sqrt{\pi N k}}{2\sqrt{L}} \quad . \qquad (2.21)$$

This result, which is more accurate than Eq. (2.18) for large N because it uses $\sum_{n=1}^{N} y_n$ rather than $\sum_{n=1}^{\infty} y_n e^{-n/N}$, suggests that Eq. (2.18) can be made more accurate by replacing the factor $gN/\pi L$ by $gN/4L$ to obtain

$$N Z_o Y(k) \cong \frac{(1 + j)\ \pi a \sqrt{\pi k}}{\sqrt{g}} \left[1 + \frac{\sqrt{gN}}{2\sqrt{L}}\right], \text{ large ka.} \quad (2.22)$$

This surprising result predicts that the impedance will vary as $N^{1/2}$ once $N > L/g$, and that the transition to the periodic result in Eq. (2.17) takes place when $N > ka^2/L$.

Finally, we can obtain a reuslt which properly contains both limits by converting the sum over s in Eq. (2.16) to an integral over j_s with a lower limit on j_s chosen to retain the relation $\sum_{s=1}^{\infty} j_s^{-2} = 1/4$. In this way we obtain the relation

$$N Z_o Y(k) \cong F_o(k) + \alpha \sqrt{N - 1} \tan^{-1} \frac{\alpha}{2\sqrt{N}}, \quad (2.23)$$

with

$$\alpha = \frac{(1 + j)\ a \sqrt{\pi k}}{\sqrt{L}}, \quad (2.24)$$

which can easily be seen to give the limit in Eq. (2.17) as $N >> ka^2/L$ and the limit in Eq. (2.22) for $1 << N << ka^2/L$. The change to N-1 in Eq. (2.23) is made to give the correct limit when N=1.

We have repeated the analysis for a small obstacle, that is where $kg \sim 1$ even though $kL >> 1$. The entire analysis and final result in Eq. (2.23) are unchanged, except that $F_o(k)$ is now the actual single obstacle admittance. In the case $kg << 1$, Gluckstern and Neri[6] have shown that

$$F_o(k) \cong 2\pi ka \left[- \frac{j}{k^2 \Delta} + \sum_{s=1}^{\infty} \frac{e^{-jb_s g/a}}{b_s} + j \frac{2\ell n 2}{\pi}\right], \quad (2.25)$$

where Δ is the cross sectional area of the (small) pillbox.

Discussion

Equation (2.23) gives a result for the average impedance (admittance) for N equally spaced identical cavities at high frequency. The transition to the periodic result shows clearly when $NL >> ka^2$. In addition, Eq. (2.23) predicts that, for $ka^2 >> NL$ the impedance will return to a $k^{-1/2}$ dependence at high frequency, but with a coefficient which varies as $N^{1/2}$ for large N, as given in Eq. (2.22). This has important implications where there are a large number of obstacles, and where conventional wisdom has up to now been to add impedances. We have checked this result by evaluating y_n numerically from Eq. (2.19). In addition, we have allowed g/L and L to be different for each cavity and confirm numerically that the $N^{1/2}$ result does not depend on delicate phase cancellations. Moreover, we expect that the analysis for the transverse coupling impedance will be parallel, and therefore believe that our conclusions are correct at high frequency for multiple obstacles of any shape in a beam pipe of any cross section.

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References

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