A SIMPLE ESTIMATE OF BUNCH ROTATION

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Abstract

The aim of this paper is to study a bunch rotation under a longitudinal space charge effect. Assuming the longitudinal space charge force to be linear, we can get some analytical expressions, which would be useful for understanding the compression of a bunch with a high current.

I. Introduction

The method of bunch rotation will be used for the 1 GeV Compressor/Stretcher Ring of the JHP (the Japanese Hadron Project) in order to compress a long proton bunch of 200 nsec up to a few tens nsec. In this method, high RF voltage is suddenly applied to rotate the bunch in the longitudinal phase space (bunch length $\propto 1/V_{RF}^{1/2}$, whereas $\propto 1/V_{RF}^{1/4}$ for adiabatic compression). With the medium energy of 1 GeV, the peak current in the bunch would be more than 100 A, so that the space charge effect becomes significant.

Computer simulations have already been made for the creation of a short proton pulse by the bunch rotation method, taking into account the strong effect of longitudinal space charge [1, 2]. However, if we get simple formulae to evaluate the space charge effect, these formulae will not only serve as a guide line for the time-consuming computer simulation, but also help understand the general properties on the space charge effect.

In this paper, presented are some analytical formulae to estimate the bunch rotation with a longitudinal space charge effect. To obtain the formulae, we use a simple model, in which the space charge force is assumed to be linear. First we will give an elementary derivation of the envelope equation including a self-force. Using the equation we will then obtain some formulae, and finally present numerical examples. However, any transverse space charge effect, probably even stronger than the longitudinal one for the JHP, is not considered in this paper.

II. Envelope Equation

In this section, we derive a general envelope equation with a self-force. Here we assume that the external force acting on a particle as well as the self-force is linear. Then the equation of motion of a particle in an ensemble (i. e., a beam) may be given as,

$$x'' + Kx = f(a)\,x\ , \qquad (1)$$

where the right hand side represents the linear self-force, and f(a) is its coefficient with a as the envelope of the beam. In the same manner as the treatment of betatron oscillation, we would write x as $x = \omega e^{i\psi}$. Then from Eq. (1) we have the amplitude equation of the particle,

$$\omega'' + K\omega - \frac{\varepsilon_0^2}{\omega^3} = f(a)\omega\ , \qquad (2)$$

where $\varepsilon_0 = \omega^2\psi'$ = constant, and this may be called emittance. Putting $\omega = f{\cdot}a$ with f being a constant, we obtain the following envelope equation;

$$a'' + \omega_0^2 a - \frac{\varepsilon_0^2}{a^3} - \frac{p_0}{a^2} = 0\ , \qquad (3)$$

where ε_0/f^2, namely the emittance of a particle on the envelope, is replaced by ε_0, and we put $K = \omega_0^2$, $f(a) = p_0/a^3$. From Eq. (3), we can immediately get the first integral of motion,

$$\frac{a'^2}{2} + \frac{\omega_0^2 a^2}{2} + \frac{\varepsilon_0^2}{2a^2} + \frac{p_0}{a} = E\ (= constant)\ . \qquad (4)$$

The E in Eq. (4) may be expressed by the initial condition. We can also define the potential energy U(a) as,

$$U(a) = \frac{\omega_0^2 a^2}{2} + \frac{\varepsilon_0^2}{2a^2} + \frac{p_0}{a}\ . \qquad (5)$$

The envelope equation thus becomes equivalent to the equation of motion for a potential in classical dynamics. The second and third terms in Eq. (5) may represent a repulsive potential due to angular momentum and a coulomb potential in classical dynamics. From Eq. (5), the time elapsed from a_0 to a_f (the lower limit of allowable a) is given by,

$$T = \int_{a_f}^{a_0} \frac{da}{a'} = \int_{a_f}^{a_0} \frac{da}{\sqrt{2(E - U(a))}}\ . \qquad (6)$$

III. Envelope Equation with Space Charge Effect

Here we study the longitudinal envelope equation only with a linear RF voltage, so that the amplitude of phase oscillation must be actually small for the equation to be valid. Since the longitudinal force of space charge is also assumed to be linear, the density distribution must be parabolic. Hence the voltage induced by direct space charge effect for perfectly conducting circular vacuum chamber is given as,

$$V_{s.c.} = 3\pi \frac{gZ_0}{2\beta\gamma^2} h^2 I_b \frac{\varphi}{\varphi_0^3} = 3\pi\left(\frac{Z_{//}}{n}\right) h^2 I_b \frac{\varphi}{\varphi_0^3}\ ,$$
$$\left(\frac{Z_{//}}{n}\right) = \frac{gZ_0}{2\beta\gamma^2}\ , \quad \left(g = 1 + 2\ln\frac{b}{a}\right)\ , \qquad (7)$$

where b is the radius of the vacuum chamber, a the beam radius, Z_0 the characteristic impedance of vacuum, $(Z_{//}/n)$ the longitudinal coupling impedance due only to the direct space charge effect, φ the RF phase and φ_0 the half bunch length in RF phase, and the others have the usual meanings. Using the Eq. (7), we can obtain the envelope equation with space charge effect as follows;

$$a'' + a - \frac{\varepsilon^2}{a^3} - \frac{p}{a^2} = 0 \qquad (8)$$

where we put $a = \varphi$ and the derivative is taken with respect to $\omega_s t$ with ω_s the synchrotron frequency. The constants ε^2 and p are,

$$\varepsilon^2 = 2\pi h \frac{|\eta|}{\beta^2}\frac{(E/e)}{V_{RF}}\left(\frac{\Delta E}{E}\right)^2 (\Delta\varphi)^2, \quad p = 3\pi\frac{(Z_{//}/n)h^2 I_b}{V_{RF}}\ , \qquad (9)$$

where $\Delta E/E$ and $\Delta\varphi$ are the half energy spread and half phase width of the beam, and the others are the usual meanings. The first constant is a dimensionless longitudinal emittance and the second one is a dimensionless space charge effect. The envelope equation obtained here is equivalent to the ones given in Refs. 2 and 3.

Some Formulae

In the following, a_0 denotes the initial value of a, and for brevity a_0' (the initial value of a') is assumed to be zero.

T_{f0}, minimum half pulse width and $(\Delta E/E)_{f0}$ without space charge effect

The minimum of the half pulse width attained by bunch rotation is,

$$T_{f0} = \frac{(2\pi)^{1/2}}{\beta\Omega}|\eta|^{1/2}\frac{(E/e)^{1/2}}{(hV_{RF})^{1/2}}\left(\frac{\Delta E}{E}\right)_i \propto \left(\frac{\Delta E}{E}\right)_i/(h^{1/2}V_{RF}^{1/2})\ , \qquad (10)$$

where Ω is 2π times the revolution frequency of the ring, and the subscripts i and f denote the initial and final values, respectively. At the time when the minimum is reached, the energy spread becomes,

$$(\Delta E/E)_{f0} = \frac{\beta\Omega T_i (hV_{RF})^{1/2}}{(2\pi)^{1/2}|\eta|^{1/2}(E/e)^{1/2}} \propto h^{1/2}V_{RF}^{1/2}\, T_i\ , \qquad (11)$$

where T_i is the initial value of half pulse width.

a_f minimum phase width (half width) with space charge effect

As the equation of E = U(a) is a fourth order with respect to a, the exact value of a_f may be obtained by the Ferrari's method which is suitable for numerical calculation. Yet certain approximate expressions would often be more useful to clarify the degree of the space charge effect. The approximate formula of a_f may be obtained by either iterative or averaging method (see Appendix I). An iterative method gives an expression of the second order approximation with respect to p,

$$a_f = a_1 + \frac{p}{a_0(a_0 + a_1)} + \frac{p^2}{2}\frac{(a_0 - a_1)}{(a_0 + a_1)^3 a_0^2 a_1}\ , \qquad (12)$$

where a_1 is a_f without space charge effect.

T compressing time

The time taken to compress the pulse to its minimum can be numerically calculated using Eq. (6); for example, when calculated by the Mori-Takahashi Method (a double exponential method), it gives an extremely accurate value. On the other hand, this compressing time can be analytically expressed as,

$$\omega_s T = \frac{2K(k)}{\sqrt{(a-c)(b-d)}}\left[a + (a-d)\left(\frac{sn(F,k')\,cn(F,k')}{dn(F,k')}\right) \cdot \left(Z(F,k') + \frac{\pi F}{2K(k)K'(k)} - \frac{dn(F,k')\,sn(F,k')}{cn(F,k')}\right)\right], \quad (13)$$

where a, b, c and d are the roots of E = U(a) in the descending order, and k is given by,

$$k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}, \quad k'^2 = 1 - k^2. \quad (14)$$

The K(k) is the complete elliptic integral of the first kind, K'(k) = K(k'), and F the incomplete elliptic integral of the first kind [4],

$$F = \int_0^{\varphi_0} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2\theta}}, \quad \varphi_0 = \tan^{-1}\sqrt{\frac{a-c}{c-d}}. \quad (15)$$

And sn, cn and dn are the Jacobi's elliptic functions, and Z is the Jacobi's Zeta function. In spite of its complicate form, Eq. (13) can be evaluated easily and fast (see Appendix II).

Yet the formula of the first order approximation with respect to p may be useful and is given by (see Appendix I),

$$\omega_s T \approx \frac{\pi}{2}\left(1 + \frac{p}{4\pi B} I\right),$$
$$I = \frac{4}{B(A+B)^{1/2}}\left[A\,K(k) - (A+B)\,E(k)\right]. \quad (16)$$

where E(k) is the complete elliptic integral of the second kind, and A, B and k are,

$$A = \frac{a_0^2 + \varepsilon^2/a_0^2}{2}, \quad B = \frac{a_0^2 - \varepsilon^2/a_0^2}{2}, \quad k^2 = \frac{2B}{A+B}. \quad (17)$$

Δt allowable time lag for the beam extraction

We may use 10% growth time of the beam pulse from its minimum to estimate the allowance for the time lag at extraction. The 10% growth time ΔT may be evaluated as,

$$\omega_s \Delta t = 2\sqrt{\frac{a_f}{g'(a_f)}\left(\frac{\Delta a}{a_f}\right)}, \quad g'(a_f) = \frac{1}{a_f}\left[2E + \frac{\varepsilon^2}{a_f^2} - 3a_f^2\right], \quad (18)$$

with $\Delta a/a$ = 10%. Here g(a) is the square of da/dt.

ΔT compression time lag

In compressing the beam pulse, the nonlinearity of RF voltage would give rise to a time lag of compressing time for the tail of the beam distribution. If there were no space charge effect, the compression time lag ΔT can be approximately estimated as,

$$\frac{\Delta T}{T} = \frac{a^2}{16}. \quad (19)$$

Even with space charge effect, this expression may still be valid, because the space charge effect would not be so strong at the tail where the particle density is low and then its gradient would be gentle.

Qualitative but useful formulae, which we can obtain from the energy expression of Eq. (4), are omitted from this paper.

IV. Numerical Examples

This section presents some numerical examples for the JHP. The parameters taken here are the kinetic energy of proton E_k = 1 GeV, $|\eta|$ = 0.17597, b/a = 2, the revolution frequency = 1.5 MHz. Then $Z_{//}/n$ becomes 120.375 Ω. In all figures presented here, the harmonic number h = 1 is taken to avoid the effect of nonlinearity of RF voltage. The initial pulse width (full width) is 200 nsec for Figs. 1 to 6 and 100 nsec for Figs. 7 to 10. The RF voltage is fixed at 300 kV except Figs. 5 and 6. The approximate formulae in Sec. III give good agreement to the values of these examples.

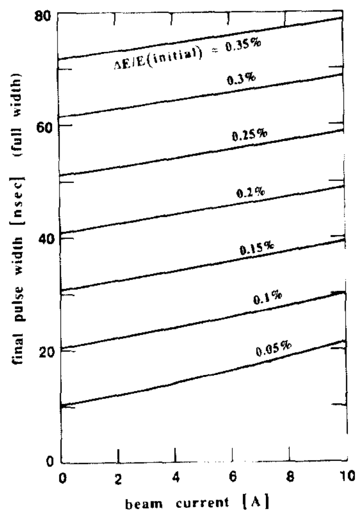

Fig. 1 Final Pulse Width versus Beam Current

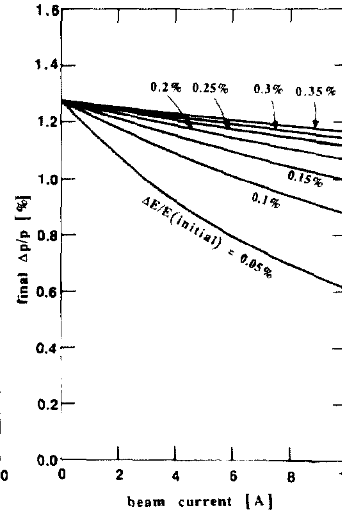

Fig. 2 Final Δp/p versus Beam Current

Acknowledgment

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[5] P. Henrici, *Computational Analysis with the HP-25 Pocket Calculator*, John Wiley&Sons, 1977.

Appendix I. Averaging Method for T and a_f

We apply here an averaging method to the envelope equation to find out approximate solutions for T and a_f. First we consider the case of p=0. By putting $a = \sqrt{\varepsilon\beta}$ in the same manner as Courant-Snyder's treatment for betatron oscillation, the square of a becomes,

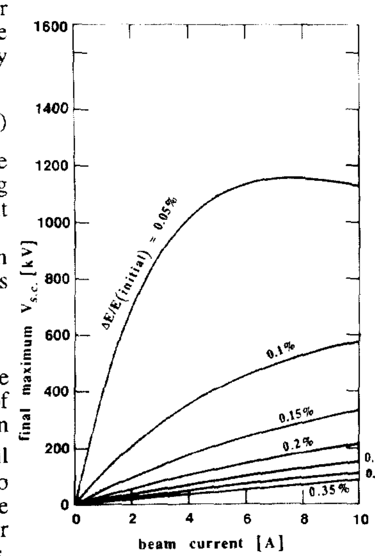

Fig. 3 Final Maximum Space Charge Voltage in Beam versus Beam Current

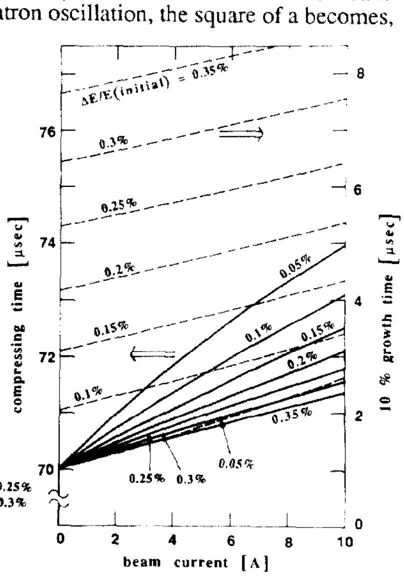

Fig. 4 Compressing Time and 10% Growth Time versus Beam Current

$$a^2 = \varepsilon\beta = A + B\cos 2\theta \quad (\theta = \omega_s t), \tag{I-1}$$

where $a'_0 = 0$ is assumed, and A and B are those of Eq. (17). For $p \neq 0$, then, the expression for a may be written as,

$$a^2 = A(\theta) + B(\theta)\cos\left(2\theta + 2\varphi(\theta)\right), \tag{I-2}$$

where $A(\theta)$, $B(\theta)$ $\varphi(\theta)$ are slowly varying variables. As there are three independent variables, we must impose other two constraints to obtain the solution. For such constraints, we may take,

$$aa' = -B(\theta)\sin\left(2\theta + 2\varphi(\theta)\right), \qquad A(\theta)^2 - B(\theta)^2 = \varepsilon^2. \tag{I-3}$$

With Eqs. (I-2) and (I-3), we can obtain,

$$\frac{A'}{2} = -\frac{pB}{2a^3}\sin\theta, \quad \frac{B'}{2} = -\frac{pA}{2a^3}\sin\theta, \qquad B\varphi' = -\frac{p}{2a^3}(B + A\cos\theta). \tag{I-4}$$

Up until here no approximation is made. Now we may average Eq. (I-4) with respect to θ. The average of φ' over a cycle of oscillation is then given by,

$$\bar{\varphi} = -\frac{p}{4\pi B}\int_0^{2\pi}\frac{B + A\cos u}{(A + B\cos u)^{3/2}}du, \tag{I-5}$$

with A and B being constant. Hence the T, one fourth of a period, can be obtained as in Eq. (16). In the same way, the changes in A and B during T are obtained as,

$$\Delta A = -\frac{Bp}{2}\int_0^{\pi}\frac{\sin\phi\, d\phi}{(A + B\cos\phi)^{3/2}} = -\frac{Bp}{2}J, \quad \Delta B = -\frac{Ap}{2}J,$$
$$J = \frac{2}{B}\left[\frac{1}{\sqrt{A-B}} - \frac{1}{\sqrt{A+B}}\right] = \frac{4}{(a_0 + a_1)\, a_0\, a_1}. \tag{I-6}$$

Then a_f for $p \neq 0$ becomes,

$$a_f / a_1 = 1 + \frac{p}{(a_0 + a_1)\, a_0\, a_1}. \tag{I-7}$$

This expression is the same as the first order term for p in Eq. (12).

Appendix II Easy and Fast Calculation of Elliptic Functions

The calculation algorithm described here is very simple and fast and yet very accurate. This is given in Refs. 4 and 5, but for convenience cited here.

Step 1: set k ($\neq 1$) and φ_0, and put $S = k^2$ and $Z = 0$. Step 2: $a_0 = 1$, $b_0 = k' = \sqrt{1 - k^2}$. Step 3: calculate next arithmetic and geometric averages,

$$a_n = (a_{n-1} + b_{n-1})/2, \quad b_n = \sqrt{a_{n-1} b_{n-1}},$$
$$c_n = (a_{n-1} - b_{n-1})/2, \quad S = S + 2^n c_n^2.$$

Step 4:

$$\Delta\varphi = \tan^{-1}\left(\frac{b_{n-1}}{a_{n-1}}\tan\varphi_{n-1}\right), \quad \varphi_n = \varphi_{n-1} + \Delta\varphi, \quad Z = Z + c_n \sin\varphi_n.$$

Step 5: if c_n is not small enough, go to Step 3.

Detail and Note for Step 4: any computer may turn out Δφ in $-\pi/2 < \Delta\varphi < \pi/2$. Put $\Delta\varphi = \Delta\varphi + \varepsilon$ with ε a very small number to avoid numerical error at specific values of φ_0 and k. If $\Delta\varphi < 0$ then put $\Delta\varphi = \Delta\varphi + \pi$, and further put $\Delta\varphi = \Delta\varphi + \pi\cdot m$, where m = integral part of φ_{n-1}/π. Moreover, with ε_1 (>0) a small number, if $\pm(\varphi_{n-1} - \Delta\varphi) > \pi/2 + \varepsilon_1$ then $\Delta\varphi = \Delta\varphi \pm \pi$.

In almost all cases, several iterations, often less than five iterations, are enough to converge. With the N-th iteration converged, the elliptic integrals and functions are then given as follows;

$$K(k) = \frac{\pi}{2a_N}, \quad E(k) = \left(1 - \frac{1}{2}S\right)K(k), \quad F(\varphi_0) = \frac{\varphi_N}{2^N a_N}, \quad Z(\varphi_0) = Z.$$

If necessary, the incomplete elliptic function of the second kind E(u) can be calculated using a relation,

$$Z(u) = E(u) - \frac{E(k)}{K(k)}F(u) \tag{II-1}$$

If φ_0 is used instead of F, the Jacobi's elliptic functions reduce to elementary functions,

$$sn(F,k') = \sin\varphi_0, \quad cn(F,k') = \cos\varphi_0,$$
$$dn(F,k') = \sqrt{1 - k^2 sn^2(F,k')} = \sqrt{1 - k^2\sin^2\varphi_0}. \tag{II-2}$$

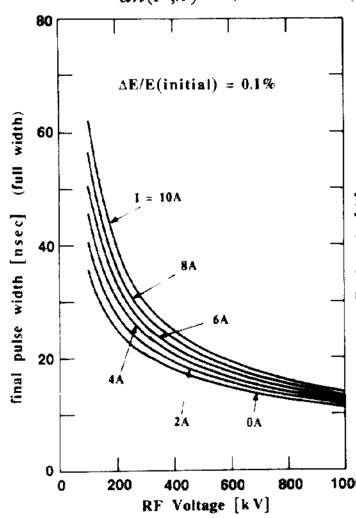

Fig. 5 Final Pulse Width versus RF Voltage

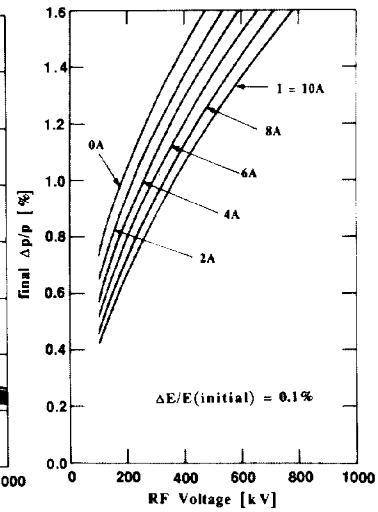

Fig. 6 Final Δp/p versus RF Voltage

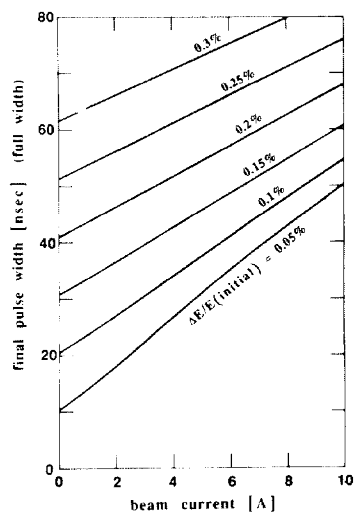

Fig. 7 Final Pulse Width versus Beam Current

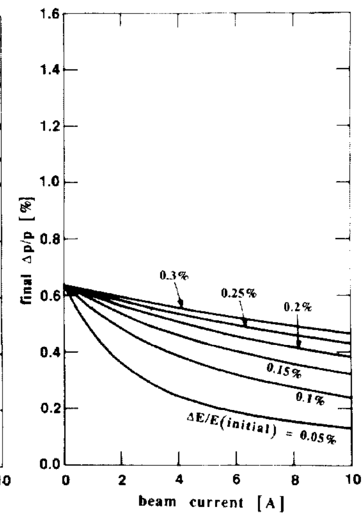

Fig. 8 Final Δp/p versus Beam Current

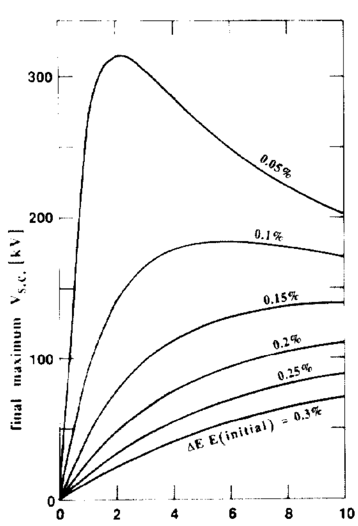

Fig. 9 Final Maximum Space Charge Voltage in Beam versus Beam Current

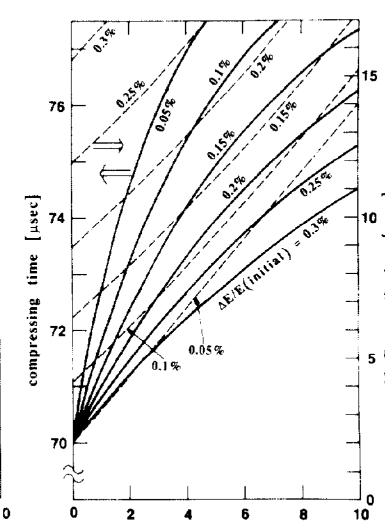

Fig. 10 Compressing Time and 10% Growth Time versus Beam Current