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EMITTANCE GROWTH, TUNE SHIFT, AND THE BUNCHED-BEAM, BUNCHED-BEAM INTERACTION*

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Abstract

While it is understood that the main limitation of beam lifetime in the Relativistic Heavy Ion Collider (RHIC) at Brookhaven is the emittance growth due to intra-beam scattering, it is important to evaluate and understand both the emittance growth and nature of the tune shift due to multiple beam-beam crossings of the bunched-beams of heavy ions. We note within RHIC, fully stripped ^{197}Au ions (charge 79e$^+$) will survive up to ten hours in the collider, with six beam crossings per revolution. With this motivation, we have developed a fully relativistic theory of both the averaged emittance growth and the averaged tune shift for the bunched-beam, bunched-beam interaction that is based on a convolution integral over the densities of the two interpenetrating bunches. In order to calculate this integral, we choose to work in a frame where one bunch of the collider is stationary, and the other is highly relativistic. This frame has the additional advantage that the microscopic heavy ion interaction becomes perpendicular in nature. In this frame the convolution integral acquires many simplifying and physically interesting features.

Introduction

1.1 Motivation

The overall problem of studying the beam-beam interaction in colliders continues to be of great interest. Over the years, many models have been proposed for the well known tune shift due to beam crossing of infinitely long beams [1–3], or bunched beams [4,5]. Of particular current interest is the beam-beam instability problem due to many beam crossings [3].

In this paper we present the foundations of a microscopic model [6] of the bunched-beam, bunched-beam interaction, where this model compliments many of the results derived earlier. Preliminary results from this model will be presented here, where these include both the emittance growth and the tune-shift due to bunched-beam, bunched-beam crossing. The preliminary results outlined here indicate the usefulness of the microscopic approach, and suggest that our more general approach to the beam-beam interaction could be quite useful in tackling the more complicated beam-beam instability problems.

1.2 Microscopic Interaction

Our model respects the form of the relativistic Lienard-Wiechert potential [7] between two charged particles during beam crossing. In general, this interaction is quite awkward to work with because both the velocity and acceleration of a single particle are required to find the fields between the two particles of interest. To overcome this problem, we transform to a frame where one of the bunches is at rest and the other bunch is highly relativistic. The particles in the highly relativistic bunch are weakly deflected by the fields of particles in the bunch at rest, and thus the electric field obtained from the Lienard-Wiechert potentials can be reduced to a more transparent and constant velocity form [6]. In this new frame the highly relativistic electric field is mainly perpendicular in nature, and is given by

$$E_\perp = \frac{Ze^2}{R^2}\frac{2\gamma^2-1}{g} \tag{1}$$

$$g = \left\{1 + \left[\frac{(2\gamma^2-1)\beta ct}{R}\right]^2\right\}^{3/2} \tag{2}$$

where Ze is the charge of the particles (identical particle scattering), R is the distance between the particles, γ is the Lorentz factor and β is the velocity of the beams, t is the time and g gives the time dependence of the interaction. The perpendicular component of the momentum transfer felt by a particle in the rest bunch can be readily calculated from Eqs. (1) and (2). In this way the scattering angle between the two particles is calculated to be:

$$\theta_{nn'} = \frac{2Z^2}{A}\frac{1}{\beta(2\gamma^2-1)}\frac{r_0}{R} \cong \frac{Z^2}{A}\frac{r_0}{\gamma^2 R} \tag{3}$$

where A is the atomic number of the charged nucleus, r_0 is the classical radius of the proton and n and n' labels the individual particles in the fixed and moving bunches respectively.

1.3 Macroscopic Quantities of Interest

In our approach, we first relate the macroscopic quantity of interest to the two body microscopic interactions that give rise to an average emittance growth or the average tune shift. Our model assumes that the two-body scattering angle $\theta_{nn'}$ in Eq.(3) can be directly equated with a change in the particles x', y' momenta. Without loss of generality, we restrict ourselves to the x-direction only. Utilizing the definition of emittance given by the Courant-Snyder invariant [8], the change in normalized emittance due to a two particle interaction can be written as,

$$\Delta E_x = 2\sqrt{\pi\gamma\beta\beta_x^* E_x}\,\sin\varphi_x \Delta x' + \pi\gamma\beta\beta_x^*[\Delta x']^2 \tag{4}$$

where β_x^* is the beta function where the two beams cross (taken to be a constant), φ_x is the angle variable of the particle, $\Delta x'$ is the projection of $\theta_{nn'}$ in the collider frame to the x-direction, and E_x is the normalized emittance. The averaged macroscopic emittance growth is found to be a double summation over all the two body scattering angles projected in the x (or y) direction, and is given by,

$$\langle \Delta E_x \rangle = \frac{\pi\gamma^3\beta\beta_x^*}{N_B}\sum_{n,n'}\theta_{nn'}^2\cos^2\xi_{nn'} \tag{5}$$

where N_B is the number of particles per bunch and $\cos\xi_{nn'}$ gives the projection of the scattering angle onto the x axis.

Additionally, the two-body tune-shift in the x (or y) direction is

$$\Delta\nu_x = \frac{\beta_x^*}{4\pi}\frac{\partial\Delta x'}{\partial x_n} \tag{6}$$

where ν_x is the x component of the particles position in the stationary bunch. In a similar manner, after a double summation, we find the average macroscopic tune-shift after beam crossing to be given by the expression,

$$\langle \Delta\nu_x \rangle = \frac{\beta_x^*}{4\pi}\gamma\frac{A}{2Z^2}\frac{\beta(2\gamma^2-1)}{r_0 N_B}\sum_{n,n'}\theta_{nn'}^2\left(1 - 2\cos^2\xi_{nn'}\right) \tag{7}$$

Evaluation of Macroscopic Quantities

2.1 Coordinate Definition

Figure 1 shows the coordinate diagram used for the calculations of the macroscopic quantities defined above. For the reasons outlined above, we prefer to transform our colliding bunches from the collider frame to a frame where one of the bunches is at rest. In this frame (shown in Fig. 1), the moving bunch has an effective Lorentz factor given by $\gamma_{eff} = 2\gamma^2 - 1$. In Fig. 1, $\vec{Z}$ is a vector

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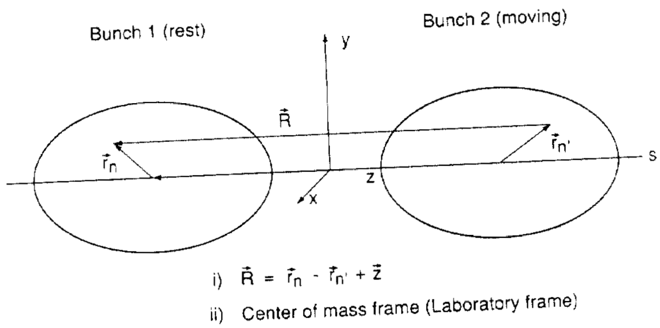

Fig. 1 Coordinate diagram.

extending from the center of the moving bunch to the center of the fixed bunch. $\vec r_{n'}$ and $\vec r_n$ are single particle coordinates in the moving bunch or the bunch at rest respectively. The coordinate $\vec R$, given in Eq.(3), is now seen to be the distance between the two particles from each bunch.

2.2 Integral Formulation

In order to evaluate the macroscopic quantities defined in Eqs. (5) and (7), we replace the double summation by a double integral over the number densities ρ_r and ρ_m for the two bunches. In this way, the averaged emittance growth is given by,

$$\langle\Delta E_x\rangle = \frac{\pi\gamma^3\beta\beta_x^*}{N_B}\int\int d^3r_n d^3r_{n'}\rho_r(\vec r_n)\rho_m(\vec r_{n'})\theta_{nn'}^2\cos^2\xi_{nn'} \qquad (8)$$

and the averaged tune shift in the x-direction is given by,

$$\langle\Delta\nu_x\rangle = \frac{\beta_x^*}{4\pi}\frac{A}{2Z^2}\frac{\beta\gamma_{eff}}{N_B r_0} \times\int\int d^3r_n d^3r_{n'}\rho_r(\vec r_n)\rho_m(\vec r_{n'})\theta_{nn'}^2\left(1-2\cos^2\xi_{nn'}\right) \qquad (9)$$

Assuming for simplicity a Gaussian form for the normalized number density, the Lorentz transformed bunch at rest in Fig. 1 has the form,

$$\rho_r(\vec r) = \frac{N_B}{\gamma\sigma\beta^*}\frac{1}{\sqrt{2\sigma_s\pi}}e^{-\pi\gamma(x^2+y^2)/\sigma\beta^*}e^{-(s-\bar s_r)^2/2\sigma_s\gamma^2} \qquad (10)$$

where s is the longitudinal coordinate, $\sigma = \sigma_x = \sigma_y$ is the average transverse normalized emittance of the beam, σ_s is related to the longitudinal extent of the bunch, and $\beta^* = \beta_x^* = \beta_y^*$. Similarly, the Gaussian number density of the moving bunch is given by the Lorentz transformed expression,

$$\rho_m(\vec r) = \frac{\gamma N_B}{\sigma B^*}\sqrt{\frac{2}{\sigma_s\pi}}e^{-\pi\gamma(x^2+y^2)/\sigma\beta^*}e^{-2\gamma^2(s-\bar s_m)^2/\sigma_s} \qquad (11)$$

where $\bar s_{r,m}$ gives the longitudinal positions of the bunch centers.

In order to deal with Eqs. (8) and (9), we express the density functions and the scattering angle in terms of their corresponding Fourier transform functions. Through convolution theory of Fourier integrals, the total number of integrations required in Eqs. (8) and (9) are halved leaving [6]

$$\langle\Delta E_x\rangle = \frac{\pi\gamma^3\beta\beta_x^*}{N_B(2\pi)^3}\int d^3K\, e^{i\vec K\cdot\vec Z}\tilde f_r(-\vec K)\tilde f_m(\vec K)\Theta_1(\vec K) \qquad (12)$$

and

$$\langle\Delta\nu_x\rangle = \frac{\beta_x^*}{4\pi}\frac{A}{2Z^2}\frac{\beta\gamma_{eff}}{r_0(2\pi)^3}\int d^3K\, e^{i\vec K\cdot\vec Z}\tilde f_r(-\vec K)\tilde f_m(\vec K)\Theta_2(\vec K) \qquad (13)$$

where

$$\tilde f_r(\vec K) = \int d^3r\, e^{-i\vec K\cdot\vec r}\rho_r(\vec r) = N_B\, e^{-\beta^*\sigma(K_x^2+K_y^2)/4\pi\gamma}e^{-K_s^2\sigma_s\gamma^4/2}e^{-iK_s\gamma\bar s_r} \qquad (14)$$

$$\tilde f_m(\vec K) = \int d^3r\, e^{-i\vec K\cdot\vec r}\rho_m(\vec r) = N_B\, e^{-\beta^*\sigma(K_x^2+K_y^2)/4\pi\gamma}e^{-K_s^2\sigma_s/8}e^{-iK_s\gamma\bar s_m} \qquad (15)$$

$$\Theta_1(\vec K) = \frac{Z^4}{A^2}\frac{r_0^2}{\gamma^4}\int d^3r\, e^{-i\vec K\cdot\vec r}\frac{x^2}{r^4} = \frac{Z^4}{A^2}\frac{r_0^2}{\gamma^4}\frac{\pi^2}{K}\left[1-\frac{K_x^2}{K^2}\right] \qquad (16)$$

$$\Theta_2(\vec K) = \frac{Z^4}{A^2}\frac{r_0^2}{\gamma^4}\int d^3r\, e^{-i\vec K\cdot\vec r}\left[\frac{1}{r^2}-\frac{x^2}{r^4}\right] = \frac{Z^4}{A^2}\frac{r_0^2}{\gamma^4}\frac{2\pi^2}{K}\frac{K_x^2}{K^2} \qquad (17)$$

2.3 Emittance Growth

The final expression for the emittance growth can be evaluated by substituting the results of the Fourier transforms (Eqs. (14–16)) into Eq. (12). Separating between longitudinal and transverse coordinates, the emittance growth is given by,

$$\langle\Delta E_x\rangle = \frac{Z^4 r_0^2\beta_x^* N_B}{8A^2\gamma}\int d^3K\, e^{i\vec K\cdot\vec Z}e^{-\beta^*\sigma(K_x^2+K_y^2)/2\pi\gamma} \times e^{-K_s^2\sigma_s\gamma^4/2}e^{iK_s\gamma(\bar s_r-\bar s_m)}\frac{1}{K}\left[1-\frac{K_x^2}{K^2}\right] \qquad (18)$$

At this point, another advantage of working in the frame defined in Fig. (1) can be seen. For large γ, the Gaussian representing the longitudinal coordinate can be approximated by the expression,

$$e^{-K_s^2\sigma_s\gamma^4/2} \cong \frac{1}{\gamma^2}\sqrt{\frac{2\pi}{\sigma_s}}\,\delta(K_s) \qquad (19)$$

This has the advantage that the Fourier integral in Eq. (12) for the emittance growth, is separable and readily evaluated. In this way the final expression for the emittance growth is given by,

$$\langle\Delta E_x\rangle = \frac{Z^4}{8A^2}r_0^2 N_B\gamma^{-5/2}\pi^{5/2}\sqrt{\frac{\beta^*}{\sigma\sigma_s}} \qquad (20)$$

2.4 Tune Shift

The quadrupole nature of the tune-shift interaction due to beam-crossing may be seen from Eq. (17). The projection acting on the scattering angle, $1-2\cos^2\xi_{nn'}$, leaves only the quadrupole term, $K_x^2/K^2 = \sin^2\theta_K\cos^2\varphi_K$, in its Fourier function $\Theta_2(\vec K)$. The simple trigonometric expression, $\sin^2\theta_K\cos^2\varphi_k$, can be expressed as a linear combination of spherical harmonics of order two. As expected, the monopole contribution to the tune shift doesn't appear in Eq. (17). The final expression for the Fourier Transform of the tune-shift is now given as,

$$\langle\Delta\nu_x\rangle = N_B\frac{\beta_x^*}{16\pi^2}\frac{Z^2}{A}\frac{r_0}{\gamma^2}\int d^3K\, e^{i\vec K\cdot\vec Z}e^{-\beta^*\sigma(K_x^2+K_y^2)/2\pi\gamma} \times e^{-K_s^2\sigma_s\gamma^4/2}e^{iK_s\gamma(\bar s_r-\bar s_m)}\frac{K_x^2}{K^3} \qquad (21)$$

Once again, using the delta-function approximation for the longitudinal component of the Gaussian number density (Eq. (19)), the integral may be cleanly separated into its longitudinal and transverse components. Hence, on evaluation, the average tune-shift for the bunched-beam, bunched-beam interaction is given by the expression,

$$\langle \Delta\nu_x \rangle = \frac{\beta_x^*}{16} \frac{Z^2}{A} N_B r_0 \gamma^{-5/2} \pi^{1/2} \sqrt{\frac{\beta^*}{\sigma\sigma_s}} \qquad (22)$$

This expression differs from the well known formula for tune-shift of two coasting beams [1] in two distinct ways: the presence of the longitudinal term, σ_s, which is not unexpected for a bunched-beam of finite extent; and the energy dependence given by $\gamma^{-5/2}$. In the near future, a careful study of the experimental evidence will be undertaken to better understand these results and their consequences. Also, since we are calculating the averaged tune-shift of all the particles in the beam rather than the tune-shift of a single particle, we expect some of the differences noted above.

Discussion and Plans for the Future

In this paper we have outlined the main ideas behind our microscopic approach to the bunched-beam, bunched-beam interaction. We have calculated two quantities of interest for bunched-beams, namely the average emittance growth and average tune shift due to beam-crossing.

We prefer to work in a frame where one of the colliding bunches is at rest, and the other highly relativistic. In this frame several mathematical simplifications and physical insights are possible. In particular, the awkward microscopic Lienard-Wiechert potential reduces to a manageable form, and the Lorentz contracted nature of the highly relativistic bunch allows the Gaussian nature of the longitudinal component to be replaced by a weighted delta-function. In this way the final Fourier transform is separable, and readily evaluated. This simplification is not possible for an infinitely long line charge.

Our result for the emittance growth due to multiple beam crossing shows that for top RHIC energies this effect is not of concern over a ten hour beam lifetime (six crossing points per revolution). This is true even with the factor of 1000 coming from the Z^2 / A factor for fully stripped ^{147}Au ions. More importantly, our result for the average tune-shift for bunched-beam, bunched-beam interactions contains two factors not present in the standard formula [1]. These are the longitudinal bunch size parameter, and the unexpected energy dependence. Further work is in progress to understand our result and the relation to available data.

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