# COMPARISON OF ANALYTICAL AND COMPUTATIONAL ESTIMATES OF REDUCTION IN LINEAR APERTURE IN THE PRESENCE OF SEXTIPOLE FIELD* 

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## Summary

For an easy comparison of the experted imporemment in the linear aperture when magnets are sorted according to various schemes, it is desirable to have a figure-of-merit which man be evaluated without time-consuming tracking ralculations. Such a figure-of-merit is proposed here in terms of the sextupole distortion functions introduced by T. ('ullins.' A test lattice composed of 512 dipoles with randon sextipole field component is used to calculate the figure-of-merit. and the in dicated improvement in the linear aperture is compared wilh the results from numerical tracking. As the linmar aperturn. two definitions are used, one by the SSC-CD( ${ }^{(1)}$ (smear $6.1^{\prime \prime}$ ) and the other by CERN SPS (smear-1.75\%).

## Introduction

In large superconducting synchrotrons, one of the matur causes of the aperture reduction is undombtedly the sextapole field in dipoles, which cannot be reduced below a certain valur without increasing the physical aperture of dipoles. Solting uf dipoles according to a predetermined procedure has beru proposed as a simple and economical way to reduce the harminal effect. ${ }^{2}$ In particular, the scheme proposed by R.l. Glurkstrm and S.Ohnuma ${ }^{3}$ requires measuring and sorting maguots in small groups, enough to cover the betatron phase adwame of $2 \pi$ which is usually four to six FODO eflls. The shemw hat been further refined by R.Li and Clwekstem.*

The dynamic aperture of a given lation is relatime rass to find by numerical particle tracking since it incolves a fow hundred revolutions and a simple question of partirles "pat" or "not lost". In contrast, the transverse accoptance of tho ring for a long-term (many hours) beam storage is dillirelt if not impossible because of the excessive amount of comiphting time required even with supercomputers. As an alternative answer to the question of long-term stability, a linear aperture is sometimes used in comparing the performance of different rings. The linear aperture is defined in terms of "smear": a measure of the deviation of transverse betatron osrillation amplitudes from their linear (invariant) values. Since the linear aperture is defined for a definite value of the smear. ${ }^{6}$ it is necessary to calculate the smear as a function of hetatron os cillation amplitude with tracking, again a time-consmming task.

An analytical expression of the smear in trems of the distortion functions of T.Collins is proposed here for a quith comparison of the effectiveness of varions sorting srhemen if dipoles. The hope is to sift through many different armange ments without testing each case with plaborate tracking calculations.

Analytical Expression of the Proposed Figure-of- Merit
Define

$$
R_{i}=\sqrt{A_{x i}^{2}+A_{y i}^{2}}
$$

where

$$
\left.\left.A_{x}^{2}=\frac{X^{2}+\left(\alpha_{x} V+\beta_{x} X^{-1}\right)^{2}}{\beta_{x}}, \quad A_{y}^{2}=\gamma^{2}+\left(\alpha_{y}\right\}_{y}, \beta_{y}\right)^{\prime \prime}\right)^{2}
$$

with ( $X, Y$ ) the displacements, and $\left(J^{\prime \prime}, Y^{\prime}\right)$ the angle of hetatron oscillation, $(\beta, \alpha)$ the linear lattice parameters. Bowh $A_{r}$ and $A_{y}$ are invariant quantities when the oscillation is lincar. In the presence of nonlinear multipoles. we have at the $i$-th turn,

$$
A_{x i}=A_{x}+\delta A_{x i}, \quad A_{y i}-A_{y}+\delta A_{y i}
$$

and

$$
R_{i}=R+\delta R_{i}
$$

Tsing Collins' distortion functions, we find

$$
\begin{gathered}
\delta A_{x}=A_{x}^{2}\left(G_{1}+G_{3}\right)-A_{y}^{2}\left(2 \bar{G}+G_{+} \quad G_{1}\right) \\
\delta A_{y}=-2 A_{x} A_{y}\left(G_{+}+G_{r}\right)
\end{gathered}
$$

where

$$
G_{\alpha}=\mathcal{A}_{\alpha} \sin \phi_{\alpha}-\mathcal{B}_{\alpha} \cos \phi_{\alpha}
$$

and

$$
\begin{gathered}
\mathcal{B}_{\alpha}\left(\varphi_{\alpha}\right)=\frac{1}{2 \sin \pi \nu_{\alpha}} \sum_{k=1}^{N} m_{\alpha} \cos \left(\mid \varphi_{\alpha} k \quad \varphi_{\alpha} \pi \nu_{\alpha}\right) \\
\mathcal{A}_{\alpha}\left(\varphi_{\alpha}\right)=\frac{d \mathcal{B}_{\alpha}\left(\varphi_{\alpha}\right)}{d \varphi_{\alpha}} .
\end{gathered}
$$

The subscript $\alpha$ represents five possible sextupole resonances and the corresponding quantities are listed in Table L. Note that the Floquet phase $\varphi$ runs from 0 to $2 \pi \nu$ while the betatron phase $\phi$ increases by $2 \pi \nu$ per turn. ${ }^{\text {i }}$

For a given arrangemeut of dipoles with soxiupole momponents, distortion functions $B_{\alpha}$ and $\mathcal{A}_{\alpha}$ are uniquely determined at each point around the ring. Since the betat ron phase $\phi_{0}$ inrreases by $2 \pi \nu$ in each turn, $G \alpha$ raries from tum for $14 m$. We define for $i=1$ to $N$ turns,

$$
\begin{equation*}
\left(\frac{\delta R}{R}\right)_{r m s}=\sqrt{\sum_{i=1}^{N} \frac{\left(R_{i}\left(\phi_{0}\right)--R\right)^{2}}{\lambda} R . ~} \tag{1}
\end{equation*}
$$

[^0]Table 1: $S \equiv B^{\prime \prime} /(2 B \rho), \phi \equiv \varphi \cdots \phi_{1}($ initial phasn $)$.

| $\alpha$ | Floque: phase $\varphi_{\alpha}$ | Betatron phase $\phi_{\alpha}$ | tune $\nu_{0}$ | Streneth mor |
| :---: | :---: | :---: | :---: | :---: |
| 1 | $\varphi_{x}$ | $\phi_{x}$ | $\nu_{*}$ | Stis ${ }_{r}^{\text {a }}$ |
| 2 | $3 \varphi_{x}$ | $3 \phi_{x}$ | $3 \nu_{x}$ | Stser ${ }^{3}$ |
| 3 | $\varphi_{x}$ | $\phi_{x}$ | $\nu_{x}$ | $S^{3} \beta_{x}^{\frac{1}{2} \beta_{y}+4}$ |
| 4 | $2 \varphi_{y}+\varphi_{x}$ | $2 \phi_{y}+\phi_{x}$ | $2 \nu_{y}+\nu_{\pi}$ | $S B_{x}^{\frac{1}{2}} B_{y} 4$ |
| 5 | $2 \varphi_{y}-\varphi_{x}$ | $2 \phi_{y} \quad \phi_{x}$ | $2 \nu_{y}-\nu_{x}$ | $S \beta_{x}^{\frac{1}{2}} \beta_{y} / 4$ |

where

$$
\bar{R}=\sum_{i=1}^{N} R_{i} / N
$$

From the definition of $R_{i}$, we find

$$
\begin{equation*}
\left(\frac{\delta R}{R}\right)_{r m s}=\sqrt{A_{x}^{2}\left(\delta A_{x i}\right)^{2}+A_{y}^{2}\left(\delta A_{y i}\right)^{2}} / R^{2} . \tag{2}
\end{equation*}
$$

Averaging over $\phi_{0}$ which will be distributed unifomly letween 0 to $2 \pi$,

$$
\left(\frac{\delta R}{R}\right)_{r m s}=\sqrt{P\left(\frac{A_{z}}{R}\right)^{4}+Q\left(\frac{A_{y}}{R}\right)^{4}-2 V \cdot\left(\frac{A_{m} A_{y}}{R^{2}}\right)^{2} A_{x} / R^{2},}
$$

where

$$
\begin{gathered}
P=\frac{1}{2}\left(\mathcal{A}_{1}^{2}+\mathcal{B}_{1}^{2}+\mathcal{A}_{3}^{2}+\mathcal{B}_{3}^{2}\right), \\
Q=2\left(\bar{A}^{2}-\overline{\mathcal{B}}^{2}\right)+\frac{5}{2}\left(\mathcal{A}_{+}^{2}+\mathcal{B}_{-}^{2}+\mathcal{A}_{-}^{2}+\mathcal{B}^{2}\right), \\
V=\left(\mathcal{A}_{1} \overline{\mathcal{A}}+\mathcal{B}_{1} \bar{B}\right) .
\end{gathered}
$$

For $A_{x}=A_{y}$, our figure-of-merit is

$$
\begin{equation*}
\left(\frac{\delta R}{R}\right)_{r m}=\sqrt{\frac{1}{4}(P+Q-2 V)} A_{t} \tag{3}
\end{equation*}
$$

If $\beta_{x}=\beta_{y}=$ const and $\nu_{x}=\nu_{y}$, this quantity, when averaged over the entire ring, reduces to the figure-of-merit used by Li and Gluckstern. ${ }^{4}$ The figure-of-merit, Eq.(3), used in this study corresponds to the smear at one particular loration of the ring.

## Test Lattice

Our test latice has 16 FODO cells followed by an insertion in each quadrant. The structure in rach coll taken the form $(\mathrm{QF}(-\mathrm{k})-\mathrm{L}-\mathrm{QD}(-2 \mathrm{k})-\mathrm{L}-\mathrm{QF}(+\mathrm{k})$ with Q and QD both a thin lens. The phase adrance in earl, ell is $90^{\circ}$ (both horizontal and vertical), $L$ is a drift space of $2 x^{2} 11$. There are dipoles with normol sextupole error betwern grads. the distance between two adjacent dipole centers is fim and it is 5 m between dipole renter and quadrupole. Altogether there are 512 dipoles and 4 insertions in the ring.

The tune excluding insertions is 16 . We adjust $\nu_{x}$ and $\nu_{y}$ by choosing the proper phase advance per insertion. We chose $\nu_{x}=17.68, \nu_{y}=18.26$, a point near the resonanee $3 \nu_{x}=54$.

## Sorting Scheme

We divide 512 magnets into 16 gorups of 32 each, one group in approximately one betatron period. Within each group, the 32 magnets are arranged according to the amombt of sextupole strength $S$. Three arrangements have been compared.

Arrangement (a); Highest $S$, 2nd lowest $S$, 3rd highest $S$. etc., then invert the scheme in every even number group, i.e. lowest $S$, 2nd highest $S$, 3rd lowest $S$, etc.. This is essentially the one suggested in ref. 3.

Arrangement (b); Highest $S,(\mathrm{~N} / 2 ; 2)$ nd highest $S, 3 \mathrm{rcl}$ highest $S,(\mathrm{~N} / 2+4)$ th highest $S, \ldots,(\mathrm{~N} / 2+1)$ si highest $S$. 2nd highest $S,(\mathrm{~N} / 2+3)$ rd highest $S$, th highest $S$. etc.. Then in every even number group, do inverting inside the first 16 magnets and the remaining 16 magnets separately.

Arrangement (c); Random arrangement(unsorted).

## Results

Both the smear and the figure-of-merit for five sets of random sextupole error fields have been calculated at the upstream end of one insertion. To find the smear, we used the tracking code TEAPOT ${ }^{8}$ which treats all magnets as thin lens. In Table 2, we list the improvement factors of the figure-of-merit and the linear aperture with two sorting schemes when they are campared with the unsorted arrangment. Note that the smear cited in ref. 6 (CERN-SPS) is $3.5 \%$, but this value should be divided by two in order to make it consistent with the SSC-CGD definition.

Table 2: The improving factor of the figure-of-merit and the linear aperture.

| Case | Sorting scheme | Analytical results | Tracking smear $6.4 \%$ | Tracking smear $1.75 \%$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | a | 5.7 | 3.3 | 3.5 |
|  | b | 13.2 | 3.0 | 3.8 |
| 2 | a | 6.4 | 3.6 | 4.2 |
|  | b | 14.0 | 2.6 | 4.3 |
| 3 | a | 3.6 | 2.1 | 1.6 |
|  | b | 13.9 | 2.0 | 1.4 |
| 4 | a | 2.7 | 4.3 | 4.1 |
|  | b | 9.8 | 4.2 | 4.6 |
| 5 | a | 3.5 | 4.2 | 4.0 |
|  | b | 7.6 | 2.7 | 4.5 |

It is clear that, for sorting scheme (a), the proposed figure -of-merit is a reliable indicator of the actual improvement in the linear aperture whether the smear is taken to be $6.4 \begin{gathered}\text { en }\end{gathered}$ $1.5 \%$. The improvement factor of the figure-of-merit tends to be somewhat larger than the numerically evaluated improwement in the linear aperture but the difference is not signifiont. As for sorting scheme (b), it is perhaps not fair to make a quantitative comparison. This scheme is based on the minimizing of distortion averaged over the entire ring whereas the smear
by tracking is evaluated at one specific foration omly. It the same time, scheme (b) ignores the strong focussing nature of the lattice (that is, the assumption $\beta_{x}=\beta_{y}$ const.) which may very well be one reason for the quantitative disagrerment. It will be interesting to see if the agrement will to improwed when the smear is also evaluated at all points in the ring.

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## References

1] T.L.Collins, "Distortion Functions", Fermilab Intermal Report 84/114, Oct. 23, 1984.
[2] L.Michelotti and S.Ohnuma, IFEE Trans. Nucl Sic.. NS-30 (1983), p. 2472.
[3] R.L.Gluckstern and S.Ohnuma, IEEE Trans. Nucl. Sci.. NS-32, No. 5, (1985), p. 2314.
[4] R.Li and R.L.Gluckstern, "Maguet Shuflling Schemes to Minimize Beam Distortion Due to Sextupole Errors", presented at the A.P.S. meeting, Baltimore, April 1988.
[5] A.Chao, "The Criteria Working Group Summary", Proceeding of the Second Advanced ICFA Beam Dynamics Workshop, Lugano, Switzerland, April 1988.
[6] A.Hilaire, "Dynamical Aperture at The SPS". Proceding of the Secomd Advanced ICFA Beam Ds wataic: Werkshep. Lugano, Switzerland, April 1988.
[7] K.-Y.Ng, "Distortion Functions", Fermilab, Internal Report FN-455, Aug. 1987.
[8] L.Schachinger and R.Talman, Particle Accelerators. 22 (1987), p. 35.


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