Scaling Relations for a Beam-Deflecting TM_{110} Mode in an Asymmetric Cavity*

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Abstract

A deflecting mode in an RF cavity caused by an aperture of the coupling hole from a waveguide is studied. If the coupling hole has a finite size, the RF modes in the cavity can be distorted. We consider the distorted mode as a sum of the accelerating mode and the deflecting mode. The finite-size coupling hole can be considered as radiating dipole sources in a closed cavity. Following the prescription given by H. Bethe,[1] the relative strength of the deflecting mode TM_{110} to the accelerating TM_{010} mode is calculated by decomposing the dipole source field into cavity eigenmodes. Scaling relations are obtained as a function of the coupling hole radius.

Introduction

A coupling between a waveguide and a cavity modifies the eigenmodes of the closed cavity. When the size of the coupling hole is finite, the modified accelerating eigenmode can have a beam-deflecting component. We study the TM_{110}-like deflecting component as a function of coupling aperture. We apply the theory of cavity coupling developed by Bethe[1] to a rectangular waveguide coupled to the side of a pillbox cavity. We assume that a TE_{10} mode in the waveguide couples to the cavity through a circular coupling hole.

The Bethe theory assumes a small coupling hole: the product of the wavenumber and the radius of the coupling hole is small compared to unity. The coupling hole acts like the radiating magnetic and electric dipoles. Applying his theory to our problem, each eigenmode of the closed cavity is driven by hypothetical electric and magnetic surface currents on the hole. Knowing the electric and magnetic current distributions responsible for the radiation, the source terms of mode-amplitude equations can be calculated, and the amplitudes of deflecting and accelerating modes can be solved numerically. We also solve amplitudes analytically by neglecting the intramode coupling. We evaluate the assumption for the analytical solution by comparing it with the numerical solution. Then, the dependence of mode amplitude on the size of the coupling hole is calculated.

Transverse Magnetic Modes in a Pillbox Cavity

We limit our scope of modes only to the Transverse Magnetic modes (TM). Assuming that the waveguide mode has a single frequency ω, the excited TM modes in the cavity depends on time as $e^{-i\omega t}$. A general TM mode in the cavity considered to be made from a waveguide by capping the ends, can be expanded with mode vectors α_k. The expansion amplitudes P_k that are to be determined give the mode strengths.

At a given oscillation frequency ω, the vanishing axial component of the electric field at the cavity wall determines the eigenvalues of γ by satisfying an equation

$$(\nabla_t^2+\gamma^2)\alpha_k(x,y)=0\ .$$

For an ideal pillbox cavity, with a length d and radius D, fields of a general TM mode can be expressed as[2]

$$E_z=\sum_{k>0}P_k\ \alpha_k(e^{ikz}+e^{-ikz})\ ,$$

$$\vec{E}_t=\sum_{k>0}P_k\ \frac{ik_f}{\gamma^2}(\nabla_t\alpha_k)(e^{ikz}-e^{-ikz})\ ,$$

$$\text{and } \vec{B}_t=\sum_{k>0}P_k\ \frac{i\mu\epsilon\omega}{c\gamma^2}\hat{z}\times(\nabla_t\alpha_k)(e^{ikz}+e^{-ikz})\ , \quad (1)$$

where $\gamma=\gamma_{mn}\equiv\frac{x_{mn}}{D}$, $k=\frac{p\pi}{d}$ with an integer p, $k_f^2\equiv\mu\epsilon\frac{\omega^2}{c^2}-\gamma_{mn}^2$ and x_{mn} is the n^{th} root of J_m.

To express the vector potentials $\vec{A}$ in terms with the eigenmodes, we define the orthogonal base vectors $\vec{a}_k^{\pm}$ as

$$\vec{a}_k^{\pm}\equiv\left[\pm\frac{ck_f}{\omega\gamma^2}(\nabla_t\alpha_k),+\frac{\alpha_k}{\frac{i\omega}{c}}\right]\ . \quad (2)$$

Then, any vector potential $(\vec{A}_\perp, A_z)$ for the TM mode can be expanded with the amplitude P_k as

$$\vec{A}=\sum_{k>0}P_k\ \vec{a}_k^+e^{ikz}+P_k\ \vec{a}_k^-e^{-ikz}\ . \quad (3)$$

We also define the normalization M_k as

$$M_k\equiv\frac{1}{k_f^2}\int\vec{a}_{\perp k}\cdot\vec{a}_{\perp k}^*dV=\left(\frac{c}{\omega\gamma^2}\right)^2\int(\nabla_t\alpha_k)(\nabla_t\alpha_k)^*dV\ , \quad (4)$$

where $\vec{a}_{\perp k}\equiv\pm\frac{ck_f}{\omega\gamma^2}(\nabla_t\alpha_k)$.

Cavity Mode Equations and the Mode Component of Electric Current

In a Lorentz gauge, the vector potential $\vec{A}$ satisfies the wave equation

$$\nabla^2\vec{A}-\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}=-\frac{4\pi}{c}\mu\vec{j}\ .$$

We expand both the vector potential $\vec{A}$ and the electric current $\vec{j}$ in terms with the cavity eigenmodes:

$$\vec{j}=\sum_k j_k\left(\vec{a}_k^+e^{ikz}+\vec{a}_k^-e^{-ikz}\right)$$

$$\text{and } \vec{A}=\sum_k P_k\left(\vec{a}_k^+e^{+ikz}+\vec{a}_k^-e^{-ikz}\right)\ ,$$

$$\text{where } \vec{a}_k^{\pm}=(\vec{a}_{k\perp}^{\pm},a_z)\ . \quad (5)$$

Substituting $\vec{A}$ and $\vec{j}$ into the wave equation, the equation for the mode coefficients P_λ is obtained as

$$\ddot{P}_\lambda+\frac{\omega_\lambda}{Q_\lambda}\dot{P}_\lambda+\omega_\lambda^2P_\lambda=4\pi cj_\lambda\ . \quad (6)$$

We included the energy dissipation effect for eigenmodes λ with cavity quality factor Q_λ and the mode eigenfrequency ω_λ.

To obtain the mode component of electric current j_k, we multiply both sides with a complex conjugate of $\vec{A}_k$ and take an inner product followed by a volume integration:

$$\int\vec{j}\cdot\vec{A}_k^*dV=j_k\int\vec{A}_k\cdot\vec{A}_k^*dV\ , \quad (7)$$

where $\vec{A}_k\equiv\frac{ck_f}{\omega\gamma^2}(\nabla_t\alpha_k)(e^{ikz}-e^{-ikz})+\hat{z}\frac{\alpha_k}{\frac{i\omega}{c}}(e^{ikz}+e^{-ikz})$.

With functions S_p and C_p defined as 1 at $p\neq 0$, $S_o\equiv 0$ and $C_o\equiv 2$, we define a normalization constant N_k as

$$N_k\equiv 2S_pM_kk_f^2+2C_p\frac{1}{(\frac{\omega}{c})^2}\int\alpha_k\alpha_k^*dV\ . \quad (8)$$

Using N_k, the mode component of the electric current is expressed as

$$j_k=\frac{1}{N_k}\int\vec{j}\cdot\vec{A}_k^*dV\ . \quad (9)$$

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Using the following relations: $\vec{B}_k = \nabla \times \vec{A}_k$, $\vec{E}_k = \frac{i\omega}{c}\vec{A}_k$, $\nabla \times \vec{E}_k = -\frac{1}{c}\frac{\partial \vec{B}_k}{\partial t} = \frac{i\omega}{c}\vec{B}_k$, and Eq. (9) can be expressed with the electric field eigenvector $\vec{E}_k$ as

$$j_k = \frac{c}{-i\omega N_k}\int \vec{j}\cdot\vec{E}_k^* dV \ . \tag{10}$$

Applying the Theory of Diffraction Developed by H. Bethe

We apply the diffraction theory developed by H. Bethe to solve the hole coupling between the waveguide and the cavity. To solve the problem, Bethe obtained a set of boundary conditions that must be satisfied on a plane at the hole. In his small-hole approximation, the fields are approximately constant over the hole opening. Bethe showed that the radiated fields are generated as if they are from an electric dipole and a magnetic dipole located at the hole. The magnetic and electric dipole moments are given as[1]

$$\vec{M} = -\frac{2}{3\pi}R^3\vec{H}_0 \quad \text{and} \quad \vec{P} = -\frac{1}{3\pi}R^3\vec{E}_0 \ . \tag{11}$$

The magnetic dipole moment points along the hole surface, and the electric dipole moment points normal to the surface. The radius of the hole is R; H_0 and E_0 are the fields at the hole in the waveguide if the hole is closed. However, if a field is present in the cavity at the hole, both H_0 and E_0 are considered as the difference between the closed waveguide field and the cavity field.

We assume a lowest TE_{10} mode is established in the rectangular waveguide. We use the magnetic field $B_d e^{-i\omega_0 t}$ of the closed waveguide to express the magnetic dipole moment M. The magnetic field B_d is measured at the small coupling hole located at the center of the waveguide end-plate, and it is parallel to the hole face. The dipole moment M in the presence of a cavity field B_{surf0} is

$$M = -\frac{2}{3\pi}R^3\left[B_d e^{-i\omega_0 t} - B_{\text{surf0}}\right] \tag{12}$$

Because there is no normal component of electric field at the hole, the electric dipole field E_{surf0}, which radiates into the waveguide from the cavity, is due to the presence of field in the cavity. The electric dipole density $\vec{p}$ is equivalent to the electric current density $\vec{j}_{equiv} = -i\omega_0\vec{p}$. The dipole moment $\vec{P}$ is given as an integrated dipole density over a volume:

$$\vec{P} = \int \vec{p}dV = \frac{1}{3\pi}R^3 E_{\text{surf0}}\ \vec{n} \ , \tag{13}$$

where $\vec{n}$ is the radial unit vector.

The magnetic dipole moment can be driven by an electric current loop. Using the definition of magnetic moment, $M(t) \equiv \int \frac{J(t)}{c}dS$, the cavity mode component of the magnetic dipole is obtained by using Eq. (10):

$$j_k = \frac{c}{N_k}(\vec{B}_k^*\cdot\vec{s})M(t) \ , \tag{14}$$

where the unit vector $\vec{s}$ points along the hole (i.e., the negative azimuthal unit vector of the cavity). Similarly, the cavity mode component of the electric dipole moment is obtained as

$$j_k = \frac{c}{N_k}\int \vec{p}\cdot\vec{E}_k^* dV = \frac{c}{N_k}(\vec{E}_k^*\cdot\vec{n})P(t) \tag{15}$$

where $\vec{n}$ is normal to the hole surface.

Pillbox TM Cavity Modes and the Mode Couplings by the Currents from Electric and Magnetic Dipoles

The bases of the TM_{mnp} modes for an ideal pillbox cavity can be defined as

$$\alpha_{mn} = iJ_m(\gamma_{mn}\rho)\sin(m\phi+\delta) \text{ for } m \neq 0 \ ,$$
$$\text{and } \alpha_{mn} = iJ_m(\gamma_{mn}\rho) \text{ for } m = 0 \ .$$

We used cylindrical coordinates with the cavity axis z, and its origin located at one end of the cavity. These base vectors require that only the imaginary component of field amplitude P_k correspond to the measurable quantities instead of the real part of complex fields.

The magnetic and electric dipole moments in the driving currents [Eq. (14) and Eq. (15)] depend on the fields present at the hole surface. However, the fields can be known only after the mode coefficients P_k are obtained. If no fields are present in the cavity, only the incident field from the waveguide drives the moments. Because the effect of currents from the electric dipole describing the radiation from the cavity mode into the waveguide is much smaller than the effect of incoming TE_{10} mode into the cavity, we neglect the electric dipole in the following argument. Expanding the surface fields in terms with mode coefficients, the current that is due to the magnetic dipole is expressed as

$$j_{km} = -\frac{c}{N_k}(\vec{B}_k^*\cdot\vec{s})\frac{2}{3\pi}R^3 \times\left(-\frac{2ka}{\pi}B_{0\text{in}}e^{-i\omega_0 t} - [P_0 B_{0\ \text{hole}} + P_{11}B_{11\ \text{hole}}]\right) . \tag{16}$$

The mode coefficients P_0 and P_{11} correspond to TM_{010} and the beam deflecting mode TM_{110}, respectively. If we define α_λ as

$$\alpha_\lambda \equiv -\frac{4\pi c^2}{N_\lambda}(\vec{B}_k^*\cdot\vec{s})\frac{2}{3\pi}R^3 \ , \tag{17}$$

the mode components of current are simplified as

$$4\pi c j_{km} = \alpha_\lambda\left(B_d e^{-i\omega_0 t} - P_0 B_{0\ \text{hole}} - P_{11}B_{11\ \text{hole}}\right) \ . \tag{18}$$

Mode Equations for a Small Bethe Hole

The mode equation drives the mode amplitudes P_k through the dipole term in the driving terms. Using the mode coupled currents in Eq. (18), the mode equation can be written as

$$\ddot{P}_0 + \frac{\omega_0}{Q_0}\dot{P}_0 + (\omega_0^2 + \alpha_0 B_0)P_0 + \alpha_0 B_{11}P_{11} = \alpha_0 B_d e^{-i\omega t}$$
$$\ddot{P}_{11} + \frac{\omega_{11}}{Q_{11}}\dot{P}_{11} + (\omega_{11}^2 + \alpha_{11}B_{11})P_{11} + \alpha_{11}B_0 P_0 = \alpha_{11}B_d e^{-i\omega t} \ , \tag{19}$$

where the waveguide frequency ω_0 is generalized to ω. We assume that the mode amplitudes P_k depend on time as $e^{-i\omega t}$. When modes are coupled, eigenfrequencies of the modes change slightly. To calculate the eigenfrequencies, we postulate no driving wave, $B_d = 0$, and a lossless cavity, i.e., $1/Q_k = 0$. Then, Eq. (19) can be solved as an eigenequation for ω_0 and ω_{11}. Using these eigenfrequencies, the complex amplitudes P_0 and P_{11} can be solved with Q_k and B_d activated in Eq. (19).

Analytical Expression of Mode Amplitudes—Neglecting Intramode Coupling

Although it is possible to solve Eq. (19) using the accelerating eigenfrequency ω_0 as ω, we can inspect the scaling relation by solving analytically for an approximate equation. We retain only the diagonal terms of Eq. (19). The approximate equation is

$$\ddot{P}_\lambda + \frac{\omega_\lambda}{Q_\lambda}\dot{P}_\lambda + \omega_\lambda^2 P_\lambda = \alpha_\lambda\left(B_d e^{-i\omega_0 t} - B_{0\ \text{surf}}\right) \tag{20}$$

The excited deflecting amplitude is calculated from

$$\ddot{P}_{11} + \frac{\omega_{11}}{Q_{11}}\dot{P}_{11} + (\omega_{11}^2 + \alpha_{11}B_{11\ \text{hole}})P_{11} = \alpha_{11}B_d e^{-i\omega_0 t} \ . \tag{21}$$

The stationary solution after a simplification with $\omega_m^2 = \omega_{11}^2 + \alpha_{11}B_{11\ \text{hole}}$ is written as

$$P_{11} = \alpha_{11}B_d\left(\frac{e^{-i\omega_0 t}}{\omega_m^2 - \omega_0^2 - i\frac{\omega_{11}\omega_0}{Q_{11}}}\right) \ . \tag{22}$$

The acceleration amplitude, whose frequency is equal to the incoming wave, is calculated from

$$\ddot{P}_0 + \frac{\omega_0}{Q_0}\dot{P}_0 + \omega_{m0}^2 P_0 = \alpha_0 B_d e^{-i\omega_0 t} \ , \quad (23)$$

where we used a definition $\omega_{m0}^2 = \omega_0^2 + \alpha_0 B_{0\ \mathrm{hole}}$. The stationary solution is written as

$$P_0 = \alpha_0 B_d \left(\frac{e^{-i\omega_0 t}}{\alpha_0 B_{0\ \mathrm{hole}} - i\frac{\omega_0^2}{Q_0}} \right) \ . \quad (24)$$

Excluding the time dependence $e^{-i\omega_0 t}$, the imaginary parts P_{11}^i and P_0^i of amplitudes are obtained from Eqs. (22) and (24)

$$P_{11}^i = \alpha_{11} B_d \frac{\frac{\omega_{11}\omega_0}{Q_{11}}}{(\omega_m^2 - \omega_0^2)^2 + \left(\frac{\omega_{11}\omega_0}{Q_{11}}\right)^2} \ , \quad (25)$$

$$\text{and } P_0^i = \alpha_0 B_d \frac{\frac{\omega_0^2}{Q_0}}{(\omega_{m0}^2 - \omega_0^2)^2 + \left(\frac{\omega_0^2}{Q_0}\right)^2} \ . \quad (26)$$

The ratio of imaginary parts of amplitudes for the excited mode P_{11} and the accelerating mode P_0 is

$$\frac{P_{11}^i}{P_0^i} = \frac{\alpha_{11}}{\alpha_0} \frac{(\alpha_0 B_{0\ \mathrm{hole}})^2 + \left(\frac{\omega_0^2}{Q_0}\right)^2}{(\omega_{11}^2 - \omega_0^2 + \alpha_{11} B_{11\ \mathrm{hole}})^2 + \left(\frac{\omega_{11}\omega_0}{Q_{11}}\right)^2} \frac{\frac{\omega_{11}}{Q_{11}}}{\frac{\omega_0}{Q_0}} \ . \quad (27)$$

Example of the Bethe Hole Radiation

The Bethe small-hole theory assumes that the product of the hole radius and the free-space wave number of the radiation is smaller than unity. For a small hole coupled to a pillbox cavity, we calculate the ratio of TM_{110} and TM_{010} amplitudes.

The transverse part of the complex magnetic fields for these modes, in which the imaginary part of amplitude v corresponds to the measurable quantity (this is due to the purely imaginary base vectors α_{mn}), can be written as

$$\vec{B}_{\perp 01} = 2(v_{01}^r + iv_{01}^i)\frac{\omega}{c\gamma_{01}} J_1(\gamma_{01}\rho)\hat{\phi} \ ,$$

$$\vec{B}_{\perp 11} = 2(v_{11}^r + iv_{11}^i)\left\{-\frac{\omega}{c\gamma_{11}^2}\right\}\left[\hat{\phi}\gamma_{11}\frac{\partial J_1(x)}{\partial x}\Big|_{x=\gamma_{11}\rho} - \frac{\vec{\rho}}{\rho} J_1(\gamma_{11}\rho)\right]$$

We assume a pillbox cavity having a gap of 32 cm, a radius of 25.3 cm, and solve Eq. (19) including the off-diagonal terms. The result shows that the amplitude expressions, Eqs. (25) and (26), are approximately correct. However, we observe some deviations. We define a characteristic quality factor Q_{chr} that makes each term in the numerator in Eq. (27) give equal contribution:

$$Q_{\mathrm{chr}} \equiv \frac{\omega_0^2}{\alpha_0 B_{0\ \mathrm{hole}}} \sim R^{-3} \ .$$

Figure 1 shows that Q_{chr} is in the normal range of cavities for the coupling hole area from 1 cm^2 to a few hundred cm^2. The result shows that the amplitudes for TM_{010} and TM_{110} modes behave like Eqs. (25) and (26) with its peak at Q_{chr}. Figure 2 shows the TM_{010} and TM_{110} amplitudes solved from Eq. (19) as a function of Q. However, the ratio of the amplitudes is a constant except at low Q (Fig. 3).

Because Q_{chr} takes a wide range at nominal size holes, behavior of individual amplitudes depends strongly on the cavity Q. At Q=5000, the amplitude of the TM_{010} mode is proportional to R^{-3} as expected from Eq. (26) (Fig. 4). But the amplitude of the TM_{110} mode is different from Eq. (25). However, the ratio of amplitudes P_{011}/P_{110} shows a clear scaling as $R^{3.01}$ at various hole sizes (Fig. 5). The TM_{010}-mode offset from the cavity axis can be calculated from the ratio of amplitudes by considering the TM_{110} mode as a perturbation. The mode offset normalized to the cavity radius is plotted against the hole area in Fig. 6, which it shows that

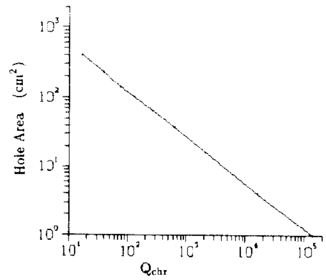

Fig. 1. The characteristic $Q_{\mathrm{chr}} = \omega_o^2/(\delta_o B_o)$ scales as R^{-3}.

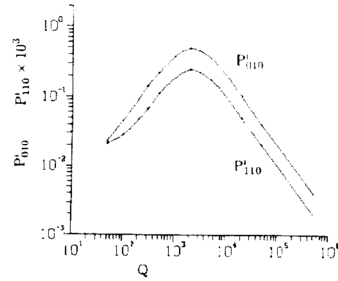

Fig. 2. Amplitude P_{010}^i and P_{110}^i, dependence on cavity Q.

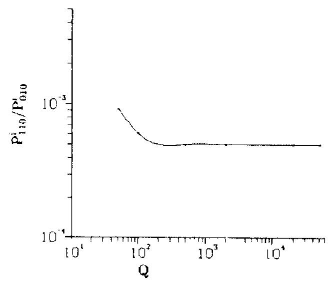

Fig. 3. The ratio of amplitude P_{110}^i/P_{010}^i does not depend on cavity Q.

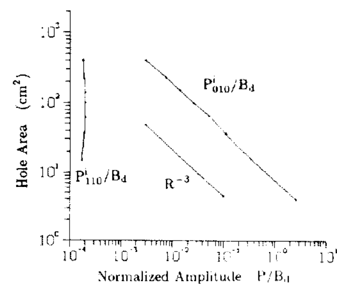

Fig. 4. A sample of amplitudes TM_{010} and TM_{110} at Q=5000.

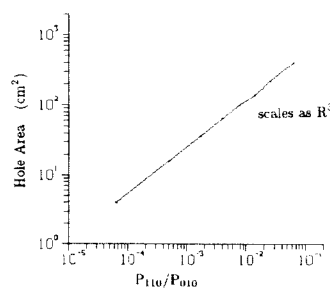

Fig. 5. The amplitude P_{110}/P_{010} is proportional to (hole radius)$^{3.0}$.

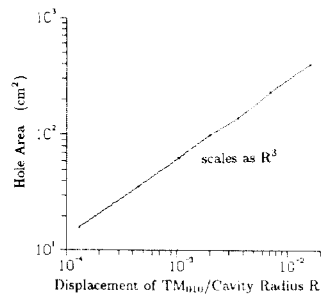

Fig. 6. The TM_{010}-mode offset is proportional to (hole radius)$^{3.0}$.

the displacement is proportional to $R^{3.0}$. We also note that the dependence for the ratio of the magnetic field is approximately the same as the accelerating mode displacement.

Conclusion

For a small hole coupling between a waveguide and a pillbox cavity, the ratio of the deflecting mode to the accelerating mode, TM_{110}/TM_{010} depends on the cubic power of the hole radius. The displacement of the accelerating mode also scales the same. The ratio of these mode amplitudes is nearly a constant over a wide range of cavity quality factor, although each amplitude depends strongly on Q. The analytic expressions of mode amplitude do not fully describe the scaling relations because the intramode coupling is not negligible.

References

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2. J. D. Jackson, *Classical Electrodynamics*, 2nd Ed. (John Wiley & Sons, NY, 1975), Sect. 8.3 and 8.11.