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SINGLE BUNCH CUMULATIVE BEAM BREAKUP*

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Introduction

The theory of cumulative beam breakup is applied to the problem of the deflection of the tail of a single bunch in a linear collider produced by the wakefields of the particles closer to the head of the bunch. An analytic expression is derived for the envelope of the bunch transverse oscillation, which can be used in estimating the magnitude of the effect.

Analysis

The difference equation for the transverse displacement $\xi(N,M)$ and angle $\Theta(N,M)$ of the Mth bunch as it enters the Nth cavity can be written as[2]

$$\xi(N+1,M) = M_{11}\xi(N,M) + \frac{\gamma_N M_{12}}{\gamma_{N+1}}[\Theta(N,M) + \phi(N,M)], \quad (1)$$

$$\Theta(N+1,M) = M_{21}\xi(N,M) + \frac{\gamma_N M_{22}}{\gamma_{N+1}}[\Theta(N,M) + \phi(N,M)], \quad (2)$$

where

$$\phi(N,M) = \frac{1}{\gamma_N}\sum_{\ell=0}^{M-1} s_{M-\ell}\, R_{N\ell}\, \xi(N,\ell) \quad ,$$

$$s_k = \frac{1}{2i}\left(e^{ik\alpha} - e^{-ik\alpha^*}\right) \quad , \quad \alpha = \omega\tau + \frac{i\omega\tau}{2Q} \quad .$$

Here $\omega/2\pi$ and Q are the frequency and quality factor of the deflecting mode, τ is the time between bunches, $mc^2\gamma_N$ is the particle energy in the Nth cavity, and $R_{N\ell}$ is a parameter proportional to the charge in the ℓth bunch and to the ratio of the shunt impedance to the Q of the Nth cavity.

The 2x2 matrix M represents the transport between cavities. In the smooth approximation, we will use a constantly focussing matrix. In this approximation one can set

$$M_{11} = M_{22} = \cos\mu_N \; , \; M_{12} = L_N \; , \; M_{21} = -\frac{\sin^2\mu_N}{L_N} \; , \quad (3)$$

where L is the distance between the centers of successive cavities.

From now on we will assume constant parameters for all cavities and a drifting beam with no acceleration, and we will drop the index N from all parameters. We can rewrite Eqs. (1) and (2) as

$$y(N+1,M) - 2\cos\mu\, y(N,M) + y(N-1,M) \cong Z(N,M) \quad , \quad (4)$$

$$Z(N,M) = r\sum_{\ell=0}^{M-1}\sin(M-\ell)\omega\tau\, \exp\left(-\frac{(M-\ell)\omega\tau}{2Q}\right)\xi(N,\ell), \quad (5)$$

where $r = RL/\gamma$. In the smooth approximation assuming that the focussing is weak, we can write $\cos\mu \cong 1 - \mu^2/2$, and Eq. (4) can be rewritten as

$$\frac{\partial^2\xi}{\partial N^2} + \mu^2\xi = Z \quad . \quad (6)$$

Equations (6) and (5) apply to successive bunches in the case of cumulative beam breakup. For single bunch beam breakup we will divide the bunch in M_T macroparticles each carrying a fraction $1/M_T$ of the total bunch charge. Equations (5) and (6) can be applied to give the deflection of successive macroparticles. Since τ is now the time increment between successive macroparticles, $\omega\tau$ can be assumed to be a small quantity. We can therefore rewrite

$$Z = r\sum_{\ell=0}^{M-1} e^{-(M-\ell)\frac{\omega\tau}{2Q}}\sin(M-\ell)\omega\tau\; \xi(N,\ell)$$

as

$$Z \cong r\,\omega\tau\sum_{\ell=1}^{M-1}(M-\ell)\;\xi(N,\ell) \quad , \quad (7)$$

where Q no longer appears explictly. In the continuous approximation, we have

$$\frac{\partial Z}{\partial M} = r\,\omega\tau\sum_{\ell=1}^{M-1}\xi(N,\ell) \quad , \quad (8)$$

$$\frac{\partial^2 Z}{\partial M^2} = r\,\omega\tau\,\xi \quad . \quad (9)$$

Equations (6) and (9) can be combined to give

$$\frac{\partial^2}{\partial M^2}\left(\frac{\partial^2\xi}{\partial N^2} + \mu^2\xi\right) = \omega\tau\, r\,\xi \quad . \quad (10)$$

The solution to Eq. (10) will be exponential in character for large M and N. We therefore try a solution of the form

$$\xi = W\exp f(N,M) \quad , \quad (11)$$

where W is a slowly varying function of N and M. If we retain only derivatives of the exponent, Eq. (10) requires that

$$\left[\left(\frac{\partial f}{\partial N}\right)^2 + \mu^2\right] \left(\frac{\partial f}{\partial M}\right)^2 = \omega\tau\ r \quad . \tag{12}$$

Using the ansatz

$$f(N,M) = -i\mu N + M^{\lambda} E(N) \quad , \tag{13}$$

we have

$$\lambda^2 (M^{\lambda-1})^2 E^2(N)[\mu^2 + (-i\mu + M^{\lambda}E'(N))^2] = \omega\tau\ r \quad .$$

Keeping only the leading term in E, we find

$$-2i\mu\ \lambda^2\ M^{3\lambda-2}\ E^2(N)\ E'(N) = \omega\tau\ r$$

from which one can derive the following equations for λ and E(N):

$$\lambda = 2/3 \quad , \tag{14}$$

$$E(N) = \frac{3}{2}\exp(i\pi/6)\left(\frac{\omega\tau\ r\ N}{\mu}\right)^{1/3} \quad . \tag{15}$$

We now need to express the exponent in Eq. (13) in terms of the physical bunch parameters. If the total bunch time is T, then the time interval between macroparticles is

$$\tau = T/M_T \quad . \tag{16}$$

The total number of actual particles in the bunch is N_T and the number of particles in a macroparticle is

$$N_p = N_T/M_T \quad . \tag{17}$$

Since the parameter r has the form

$$r = \frac{L}{\gamma}\frac{N_p e^2}{2mc}\frac{Z_\perp}{Q} \quad , \tag{18}$$

we can use Eqs. (16) to (18) to write the exponent of Eq. (11) as

$$f = -i\mu N + \frac{3}{2} F \quad , \tag{19}$$

where

$$F = e^{\frac{i\pi}{6}} \left[\frac{N_T\ e^2 \left(\frac{Z_\perp}{QL}\right) \omega T}{2mc\gamma}\right]^{1/3} \left(\frac{M}{M_T}\right)^{2/3} \frac{(NL)^{1/3}}{(\mu/L)^{1/3}} \quad . \tag{20}$$

If we define t as the time from the front of the bunch and z as the length along the structure, the exponent has the form

$$F = e^{\frac{i\pi}{6}} \left[\frac{I_p\ e \left(\frac{Z_\perp}{QL}\right)}{2mc^2\gamma}\right]^{1/3} \frac{(\omega t)^{1/3}(ct)^{1/3}z^{1/3}}{(\mu/L)^{1/3}} \quad , \tag{21}$$

where $I_p = eN_p/\tau = eN_T/T$ is the peak current, and ct is the distance from the front of the bunch. We solved the difference equations of Eqs. (1) and (2) numerically, for the following set of parameters: $r = 2 \times 10^{-5}$, $\omega\tau/2\pi = 8 \times 10^{-5}$, $Q = 3$, $M_T = 250$ and $N = 3000, 6000, 9000, 12000$. The phase advance, μ, was chosen to be $\pi/150$, so that if N is a multiple of 300 the phase factor $\exp(i\mu N)$ is 1. The results for ξ/ξ_o are plotted in Fig. 1.

According to Eqs. (11), (13), (14), and (15), ξ/ξ_o is only a function of $M^{2/3}N^{1/3}$. This is checked in Fig. 2, where the data of Fig. 1 is plotted against $M^{2/3}N^{1/3}$. The data from the four curves of Fig. 1 is seen to lie on the same universal curve, confirming the validity of Eqs. (11), (19) and (20).

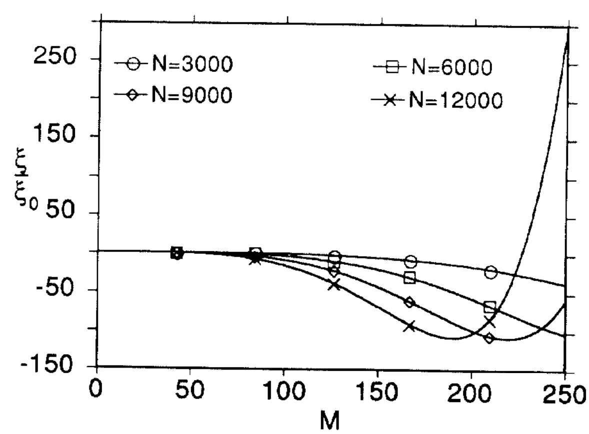

Figure 1

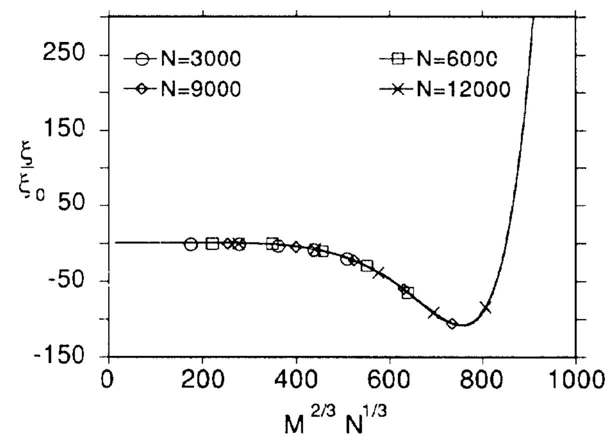

Figure 2

Discussion

Equation (21) has been derived under the assumption that only one mode is excited. In the general case Z in Eq. (5) can be written as a sum over the different modes

$$Z = \sum_k r_k \sum_{\ell=0}^{M-1} e^{-(M-\ell)\frac{\omega_k \tau}{2Q_k}} \sin(M-\ell)\omega_k\tau\ \xi(N,\ell) \quad , \tag{22}$$

where the index k labels the different modes. In the limit $\tau \to 0$ the contributions of the different modes add up coherently so Eq. (9) can be written as

$$\frac{\partial^2 z}{\partial M^2} \cong \left(\sum_k r_k\ \omega_k\right)\tau\ \xi(N,\ell) \quad . \tag{23}$$

The exponent can then be written in the general case as

$$F = e^{i\frac{\pi}{6}} \left(\frac{I_p\ e}{2mc\gamma} \sum \left(\frac{Z_{\perp k}\omega_k}{LQ_k}\right)\right)^{1/3} \frac{t^{2/3}\ z^{1/3}}{(\mu/L)^{1/3}} \quad . \tag{24}$$

Only the exponent has been derived here. A more complete saddle point calculation gives the factor W in front of the exponent in Eq. (10). The result, which we state without proof, is that the displacement of the transverse oscillation of the bunch, given a constant initial displacement ξ, is

$$\frac{\xi}{\xi_o} \cong \mathrm{Re}\left[\frac{\exp(-i\mu N + 3F/2)}{\sqrt{6\pi F}}\right] \quad , \tag{25}$$

where

$$F^3 = \frac{ie\ I_p\ t^2 z}{2mc\gamma(\mu/L)} \sum_k \left(\frac{Z_{\perp k}\ \omega_k}{LQ_k}\right) \quad . \tag{26}$$

The universal dependence of ξ/ξ_o on F (which is proportional to $M^{2/3}N^{1/3}$) in Fig. 2 is explicit in Eq. (25).

For the situation where γ and μ vary with z, we need to make the replacements

$$\frac{z}{\gamma\mu} \to \int \frac{dz}{\gamma(z)\ \mu(z)} \tag{27}$$

and

$$\mu N \to \int \mu dN \quad . \tag{28}$$

References

1. Worked supported by the Department of Energy.

2. Gluckstern, Cooper and Channell, Particle Accelerators 16, 125 (1985).