TRANSVERSE AND LONGITUDINAL COUPLED BUNCH INSTABILITIES IN TRAINS OF CLOSELY SPACED BUNCHES*

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ABSTRACT

Damping rings for the next generation of linear collider may need to contain several bunch trains within which the bunches are quite closely spaced (1 or 2 RF wavelengths). Methods are presented for studying the transverse and longitudinal coupled bunch instabilities, applicable to this problem and to other cases in which the placement of the bunches is not necessarily symmetric.

1. INTRODUCTION

The present SLAC design for a damping ring in a ~ 1 TeV linear collider using multibunching requires that it contain up to ten bunch trains, where each train may contain ten or more bunches. This paper presents methods for studying the transverse and longitudinal coupled-bunch instabilities in a ring containing bunches in arbitrary RF buckets, circulating in one direction. The n normal modes of coupled oscillation of n bunches have been studied by other authors[1–4] for the case of bunches symmetrically located on the ring circumference. We present a semi-analytic, normal-modes approach, which is quite general in that the bunches need not be symmetrically placed and the wake may contain all frequencies of interest. The problem of finding the coherent frequencies and oscillation modes amounts to finding the eigenvalues and eigenvectors of a matrix; the elements of the matrix are derived analytically. It is then straightforward to solve for the eigenvalues and eigenvectors numerically. The imaginary parts of the eigenvalues determine the long-term stability. However, even if all the normal modes are stable, interference between modes can produce large transients.

Thus it is sometimes desirable to know the motion of each bunch as a function of time, given the initial conditions of all the bunches. We have two independent methods of obtaining this information: (1) Given the coherent frequencies and normal modes, the Laplace transform can be used to obtain the motion of the bunches, taking the initial conditions correctly into account. (2) We have also used a computer tracking method in the transverse case, to obtain the offset of each bunch as a function of time. At least one of these methods is always applicable, and in many cases both of them are. Such cases were used to verify our analysis and computer codes, obtaining identical results with the two independent methods. Finally, we emphasize that the normal modes analysis seems to be reliable and practical in all regimes of interest, and if it is not necessary to know details of the transient behavior, this method is sufficient by itself.

2. NORMAL MODES ANALYSIS

We assume that there are a total of n bunches in the ring. The bunches are taken to be point macroparticles with relativistic velocity c on the design orbit $s_0(t)$.

2.1 Longitudinal motion

Suppose the orbit of a bunch undergoing rigid coherent synchrotron oscillations is $s(t)$. Then the longitudinal time displacement of the bunch away from its synchronous position is:

$$\tau(t) = \frac{s(t) - s_0(t)}{c} \quad . \tag{1}$$

Denote the deviation from the design energy E_0 by:

$$\epsilon(t) = E(t) - E_0 \quad . \tag{2}$$

Both $\tau(t)$ and $\epsilon(t)$ oscillate with synchrotron frequency ω_s, which is assumed to be much less than the design orbital angular frequency ω_0. The equations of motion for bunch i are:

$$\dot{\tau}_i = -\alpha\frac{\epsilon_i(t)}{E_0} \quad , \tag{3}$$

and

$$\dot{\epsilon}_i = \frac{1}{T_0}\left[e\hat{V}\sin(\omega_{rf}\tau_i + \phi_{0i}) - U_{rad}(\epsilon_i) + eV_w(\tau_1, ..., \tau_n, t)\right] \quad . \tag{4}$$

In the first of these equations, α is the momentum compaction factor. The first term in the equation for $\dot{\epsilon}_i$ is the energy change due to RF, the second term is the energy change due to radiation loss, and the third term is the energy change due to wake fields, all per turn, while T_0 is the orbital period.

Assuming a total of n bunches and including turn-to-turn wake field effects, we have

$$eV_w = Ne^2\sum_{j=1}^{n}\sum_{q=0}^{\infty} W(qT_0c + L_{ij} + c\tau_{j,q} - c\tau_i) \quad . \tag{5}$$

Here $W(z)$ is the wake function, which gives the voltage in the cavity (or any other high impedance structure of interest) due to a unit charge that passed through it a time $t = z/c$ ago. The units of W(z) are [V/Coul]. The time displacement of the j^{th} bunch q turns ago is denoted by $\tau_{j,q}$. The distance L_{ij} is the "equilibrium" spacing between bunches i and j, that is, their spacing when $\epsilon_k = \tau_k = \dot{\epsilon}_k = \dot{\tau}_k = 0$. We take $L_{ij} > 0$ for $i > j$, and $L_{ij} = -L_{ji}$. Note that the wake function W is zero when its argument is less than zero, i.e. when $q = 0$ and $i < j$.

Let us assume that ϵ_i and τ_i are small enough that we can expand each of the three terms contributing to the energy change to first order. If we combine the $\dot{\tau}_i$ and $\dot{\epsilon}_i$ equations to eliminate ϵ_i, we obtain:

$$\ddot{\tau}_i + \lambda_i\dot{\tau}_i + \omega_{s,i}^2\tau_i = -\frac{\alpha Ne^2c}{E_0T_0}\sum_{j=1}^{n}\sum_{q=0}^{\infty}\left(\frac{dW}{dz}\right)_{qT_0c+L_{ij}}\tau_{j,q} \quad , \tag{6}$$

where

$$\omega_{s,i}^2 = \frac{\alpha e\hat{V}\omega_{rf}\cos\phi_{0i}}{E_0T_0} - \frac{\alpha Ne^2c}{E_0T_0}\sum_{j=1}^{n}\sum_{q=0}^{\infty}\left(\frac{dW^{hom}}{dz}\right)_{qT_0c+L_{ij}} \tag{7}$$

gives the perturbed synchrotron frequency of the i^{th} bunch. The quantity W^{hom} includes only the higher order modes of the wake, since we assume there is feedback compensating the effects of the fundamental mode. The damping due to radiation is represented by:

$$\lambda_i \equiv \frac{1}{T_0}\frac{dU_{rad}}{d\epsilon_i}\bigg|_{\epsilon_i=0} \quad . \tag{8}$$

We have used the following relation between the synchronous phases ϕ_{0i} and bunch spacings L_{ij}:

$$e\hat{V}\sin\phi_{0i} - U_0 + Ne^2\sum_{j=1}^{n}\sum_{q=0}^{\infty} W(qT_0c + L_{ij}) = 0 \quad . \tag{9}$$

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We shall assume that the L_{ij} are integral numbers of RF wavelengths; that is, the presence of wake fields and synchrotron radiation loss does not cause different bunches to ride at significantly different points on the RF waveform.

Let us look for solutions in the form of normal modes

$$\tau_i(t) = a_i e^{-i\Omega t} \quad . \tag{10}$$

Here the a_i are constants, and Ω is the coherent frequency of the mode. Then the equations for the a_i's can be written

$$(\Omega^2 + i\Omega\lambda_i - \omega_{s,i}^2)a_i + \sum_{j=1}^{n} \chi_{ij}(-i\Omega)a_j = 0 \quad , \tag{11}$$

where

$$\chi_{ij}(s) \equiv -\frac{\alpha N e^2 c}{E_0 T_0} \sum_{q=0}^{\infty} \left(\frac{dW}{dz}\right)_{qT_0c+L_{ij}} e^{-qT_0 s} \quad . \tag{12}$$

We take the $q = 0$ term in this sum to be zero when $i = j$, since it is readily shown that the local wake is independent of τ_i (where τ_i is the average of τ over the bunch distribution). Substituting a wake field of the form

$$W(z) = \begin{cases} \sum_m W_m \cos(k_m z) & (z > 0) \\ 0 & (z < 0), \end{cases} \tag{13}$$

and performing the sums over q yields:

$$\begin{aligned} \chi_{ij}(s) = &-\frac{\alpha N e^2 c}{E_0 T_0} \sum_m \frac{1}{2} W_m \left[\frac{ik_m e^{ik_m L_{ij}}}{1 - e^{(ik_m c - s)T_0}} \right. \\ &\left. - \frac{ik_m^* e^{-ik_m^* L_{ij}}}{1 - e^{(-ik_m^* c - s)T_0}} \right] \qquad (i > j) \\ \chi_{ij}(s) = &-\frac{\alpha N e^2 c}{E_0 T_0} \sum_m \frac{1}{2} W_m \left[\frac{ik_m e^{ik_m L_{ij}} e^{(ik_m c - s)T_0}}{1 - e^{(ik_m c - s)T_0}} \right. \\ &\left. - \frac{ik_m^* e^{-ik_m^* L_{ij}} e^{(-ik_m^* c - s)T_0}}{1 - e^{(-ik_m^* c - s)T_0}} \right] \qquad (i \le j) \quad . \end{aligned} \tag{14}$$

Note that the k_m are complex to account for damping of the wake. Suppose that $|\Omega|$ and $\omega_{s,i}$ are close to the unperturbed synchrotron frequency ω_s (which may be taken positive or negative), so that we can approximate:

$$\Omega^2 + i\lambda_i\Omega - \omega_{s,i}^2 \approx 2\omega_s\left(\Omega - \omega_s + i\frac{\lambda_i}{2}\right) \quad . \tag{15}$$

We also replace $\chi_{ij}(-i\Omega)$ by $\chi_{ij}(-i\omega_s)$ in Eq. (11). Then we can obtain a set of n coherent frequencies Ω, each with corresponding eigenvector $\vec{\mathbf{a}}$, by solving the linear eigenvalue problem

$$\mathbf{M}\vec{\mathbf{a}} = \Omega\vec{\mathbf{a}} \quad , \tag{16}$$

where the elements of the matrix $\mathbf{M}$ are given by

$$M_{ij} = (\omega_s - i\frac{\lambda_i}{2})\delta_{ij} - \frac{\chi_{ij}(-i\omega_s)}{2\omega_s} \quad , \tag{17}$$

and $\vec{\mathbf{a}}$ is the vector $(a_1, ..., a_n)$.

2.2 Transverse motion

A modal analysis may also be carried out for the transverse case. We use the smooth focusing approximation, in which the focusing function k is the inverse of the average around the ring of the betatron function. The betatron angular frequency is then $\omega_\beta \equiv kc$, and the equation of transverse motion for transverse offset x is

$$\ddot{x}_i + \lambda\dot{x}_i + \omega_\beta^2 x_i = F_W(t) \quad , \tag{18}$$

where λ is a coherent damping parameter, and $F_W(t)$ is the force due to the transverse wake.

Although the long range transverse wake force is localized to the RF cavities (and any other high-Q structures), we may treat it as though an averaged force were distributed around the ring, provided that the coherent tune shift is small compared to 2π. That is, we take the driving term $F_W(t)$ to be:

$$F_W(t) = \frac{N e^2 c^2}{E_0} \sum_{j=1}^{n} \sum_{q=0}^{\infty} \widetilde{W}^{\perp}(L_{ij} + qT_0c)x_j(t - qT_0) \quad , \tag{19}$$

where $\widetilde{W}^{\perp}(z)$ is the wake per unit length, averaged around the circumference. For the case of a transverse wake due to, say, an RF cavity, we have:

$$\widetilde{W}^{\perp}(z) = \frac{W^{\perp}(z)}{cT_0} \quad , \tag{20}$$

where $W^{\perp}(z)$ is the transverse wake function in the cavity (units [V/Coul/m]). Thus the equation of motion is:

$$\ddot{x}_i + \lambda\dot{x}_i + \omega_\beta^2 x_i = \frac{N e^2 c}{E_0 T_0} \sum_{j=1}^{n} \sum_{q=0}^{\infty} W^{\perp}(L_{ij} + qT_0c)x_j(t - qT_0) \, . \tag{21}$$

The transverse wake function is of the form

$$W^{\perp}(z) = \begin{cases} \sum_m W_m^{\perp} \sin(k_m z) & (z > 0) \\ 0 & (z < 0). \end{cases} \tag{22}$$

Looking for normal mode solutions

$$x_i(t) = a_i e^{-i\Omega t} \quad , \tag{23}$$

we may obtain coherent frequencies and eigenmodes as in the longitudinal case, the only difference being that the χ_{ij}'s must be replaced by:

$$\begin{aligned} \chi_{ij}^{\perp}(s) = &-\frac{N e^2 c}{E_0 T_0} \sum_m \frac{1}{2} W_m^{\perp} \left[\frac{i e^{ik_m L_{ij}}}{1 - e^{(ik_m c - s)T_0}} \right. \\ &\left. - \frac{i e^{-ik_m^* L_{ij}}}{1 - e^{(-ik_m^* c - s)T_0}} \right] \qquad (i > j) \quad , \\ \chi_{ij}^{\perp}(s) = &-\frac{N e^2 c}{E_0 T_0} \sum_m \frac{1}{2} W_m^{\perp} \left[\frac{i e^{ik_m L_{ij}} e^{(ik_m c - s)T_0}}{1 - e^{(ik_m c - s)T_0}} \right. \\ &\left. - \frac{i e^{-ik_m^* L_{ij}} e^{(-ik_m^* c - s)T_0}}{1 - e^{(-ik_m^* c - s)T_0}} \right] \qquad (i \le j) \quad . \end{aligned} \tag{24}$$

3. LAPLACE TRANSFORM SOLUTION

For definiteness, we discuss the longitudinal case; the transverse case is analogous if τ is replaced by x, ω_s is replaced by ω_β, and χ_{ij} is replaced by $\chi_{ij}^{\perp}$. Defining

$$\widetilde{\tau}_i(s) \equiv \int_0^{\infty} e^{-st}\tau_i(t)dt \quad , \tag{25}$$

performing the Laplace transform on Eq. (6), and rearranging terms we obtain:

$$\sum_{j=1}^{n} \left[(s^2 + \lambda_i s + \omega_s^2)\delta_{ij} - \chi_{ij}(s)\right]\widetilde{\tau}_j(s) = (s + \lambda_i)\tau_i(0) + \dot{\tau}_i(0) \, . \tag{26}$$

Define $A_{ij}(s)$ to be the quantity in square brackets divided by $2\omega_s$. The roots of $\det \mathbf{A} = 0$ are easily seen to be $s = i\Omega_k$, where the Ω_k's are the coherent frequencies obtained in the normal modes analysis. Here we assume, consistent with that analysis, that the poles $-i\Omega_k$ are near $-i\omega_s$, so that

$$2\omega_s A_{ij} \approx [-2i\omega_s(s + i\omega_s) - i\omega_s\lambda_i]\delta_{ij} - \chi_{ij}(-i\omega_s) \quad , \tag{27}$$

from which, as one would expect, we get

$$A_{ij}(-i\Omega) = M_{ij} - \Omega\delta_{ij} \quad . \tag{28}$$

Solving Eq. (26) to obtain the $\widetilde{\tau}_i$ yields:

$$\widetilde{\tau}_i(s) = \frac{1}{2\omega_s} \sum_{j=1}^{n} \frac{c_{ji}}{\det \mathbf{A}}[(s + \lambda_j)\tau_j(0) + \dot{\tau}_j(0)] \quad , \tag{29}$$

Here $c_{ji}(s)$ is the ji^{th} cofactor of the matrix $\mathbf{A}$, that is, for $\det \mathbf{A} \neq 0$, $(c_{ji}/\det \mathbf{A}) = (\mathbf{A}^{-1})_{ij}$. Taking the inverse Laplace transform we obtain:

$$\tau_i(t) = \frac{1}{2\pi i} \int_C e^{st} \sum_{j=1}^{n} \frac{c_{ji}(s)[(s + \lambda_j)\tau_j(0) + \dot{\tau}_j(0)]}{2\omega_s(-i)^n \prod_{k=1}^{n}(s + i\Omega_k)} \quad , \tag{30}$$

where the contour C is parallel to the imaginary axis and to the right of all the poles. Closing the contour to the left and applying the residue theorem, we obtain the solutions for the offsets as a function of time:

$$\tau_i(t) = \sum_{l=1}^{n} \frac{\sum_{j=1}^{n} c_{ji}(-i\Omega_l)[(-i\Omega_l + \lambda_j)\tau_j(0) + \dot{\tau}_j(0)]}{(-2i\omega_s)\prod_{k=1, k\neq l}^{n}(-\Omega_l + \Omega_k)} e^{-i\Omega_l t} \quad . \tag{31}$$

This can be evaluated numerically once we have obtained the coherent frequencies Ω_l as discussed earlier. For very short range wakes, there can be numerical difficulties with a straightforward computation of the cofactors, because in this case we need the determinant of a matrix whose values differ by many orders of magnitude. This situation can arise for transverse wakes when strongly damped cavities[5] are used. However, when the wakes are very short range, the tracking method discussed in the next section is very efficient.

4. COMPUTER TRACKING PROGRAM

An alternate method of obtaining the $x_i(t)$'s, which we have found useful in the transverse case, is to use a simple tracking program. The motion of the bunches is divided into two parts, (1) a kick at the RF cavities (or other localized impedance):

$$x_n^+ = x_n^-$$

$$x_n'^+ = x_n'^- + \frac{Ne^2}{E}\sum_{q=0}^{q_{max}}\sum_{j=1}^{n} W^\perp((n-j)\ell + qC)x_j^{old}(q) \quad , \quad (32)$$

and (2) a mapping around the rest of the ring (assuming $\beta' = 0$ at the cavity):

$$\begin{pmatrix} x_n^- \\ x_n'^- \end{pmatrix} = e^{-\lambda C/2c}\begin{pmatrix} \cos\mu & \beta\sin\mu \\ -\frac{1}{\beta}\sin\mu & \cos\mu \end{pmatrix}\begin{pmatrix} x_n^{old\,+} \\ x_n'^{old\,+} \end{pmatrix} \quad . \quad (33)$$

Here the superscript minus (plus) on x_n and x'_n denotes the value just before (after) passing through the RF cavity, and $x_j^{old}(q)$ is the offset at the cavity of the j^{th} bunch q turns ago. Also, C is the circumference of the ring, λ is the coherent damping parameter, and μ is the coherent phase advance around the ring. The value of q_{max} must be large enough that the wake fields are negligible after q_{max} turns; therefore this method is less practical than the Laplace transform method when there are many bunches and very long range wakes.

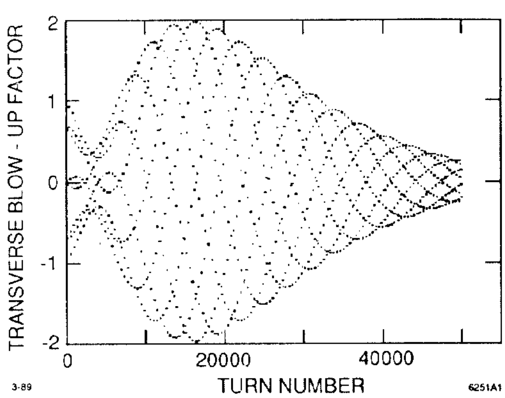

Fig. 1. Transverse displacement of the last bunch in a train of 10 bunches.

5. EXAMPLE

In Fig. 1 we show an example in which the transverse coherent frequencies are obtained, and the Laplace transform method was used to obtain the transverse offset of a bunch as a function of turn number. There are a total of 10 bunches spaced 21 cm apart in a ring of circumference 155.1 m, the average beta function is 2 m, and the coherent damping parameter is 600 sec^{-1}. The Q of all the wake frequency components was taken to be 500, for the sake of comparing the Laplace transform and tracking methods (with damped cavities, the Q's can be much lower). The results obtained using the tracking method are indistinguishable from those in the figure. Note there is transient blow-up due to interference, even though the imaginary parts of all ten coherent frequencies are negative. This transient behavior is very important in rings where storage times are only brief, for instance in damping rings. In addition, for strong wakes and long trains of bunches, this transient can be large enough to cause beam loss at injection.

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