ELEMENTARY ANALYSIS OF PHASE SPACE PAINTING

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Abstract

Analytical expressions are given in order to help understand phase space painting used in H- injection. Simple models will make it possible to analytically study both one and two dimensional paintings in the transverse planes. The longitudinal painting will be treated separately.

I. Introduction

H^- injection is becoming widely used in circular proton accelerators. This injection scheme makes it possible to inject a beam at the point of phase space already occupied by previously injected beam. Therefore an intense proton beam can be accumulated into the ring without largely increasing the beam emittance as well as the physical aperture of the ring. In addition, it is more desirable to incorporate the phase space painting into this scheme to cope with the space charge effect and to reduce the hitting probability at the stripping foil of H^- (or H^0). This combination of H^- injection and phase space painting will also be employed in the Compressor/Stretcher Ring of the JHP (Japanese Hadron Project) [1].

The aim of this paper is to help our understanding about phase space painting. Under simple assumptions, we will first obtain relations between the real space density and the phase space density for both one- and two- dimensional transverse betatron motions. In this analysis, however, the transverse beam size due to $\Delta p/p$ has not been taken into account, since it is relatively small for the JHP ring (at least until bunch rotation does work to compress a beam pulse [1]) and probably for other high intensity machines. Second we will study the phase space painting for a beam injected with a small emittance, and thereby obtain the time trajectories of injection point for various real space densities. Finally, simple analyses of the longitudinal painting will be presented.

Here we should note that from the point of view of the space charge effect the most favorable distribution in the transverse direction is a uniform distribution, while for the longitudinal direction it may be the one with a linear gradient, for which the distribution itself becomes parabolic, or may be the one proportional to the form of RF Voltage.

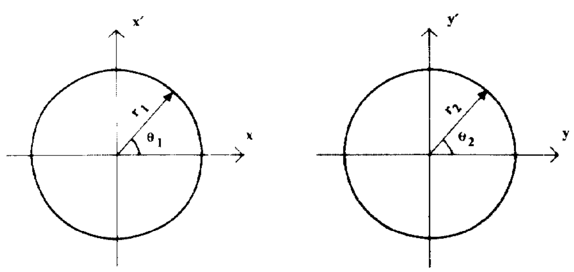

Fig. 1 Coordinates in the Normalized Phase Space

Assumptions

We put the following assumptions throughout this paper.

(1) the motion of a particle in the ring is circular on a normalized phase space of betatron or synchrotron oscillation. It just holds for the linear equation of motion. In case of synchrotron oscillation, it is only valid for a small amplitude, but not for a large amplitude. However, the following analysis may still be applicable for an amplitude not too large, since the assumption does not impose that the frequency of oscillation be independent of its amplitude.

(2) As soon as a beam has injected, its phase space distribution becomes uniform in the θ direction (Fig. 1). It would be justified by considering that any integral multiple of tune may not be commensurable with an integer, at least with a small integer, and that the tune usually has a dependence on the amplitude of oscillation in an actual ring. Since the synchrotron oscillation is relatively slow, however, it may take a longer time for the longitudinal distribution to become uniform in the θ direction.

(3) the injection point stays on the x-axis in the normalized phase space. Even if so assumed, it does not lose generality, because of the assumption of (2).

(4) the emittance of injected beam is infinitesimal. A beam injected from a linac would usually have an emittance much smaller than the beam circulating in a ring.

II. Phase Space Painting for Betatron Motion

One Dimensional Case

The density n(x) in the normalized real space has the following relation to the density ρ(x) in the normalized phase space [2, 3]:

$$n(x) = 2\int_x^a \frac{\rho(r)}{\sin\theta}\,dr = \int_x^a \frac{2\rho(r)}{\sqrt{r^2-x^2}}\,r\,dr \ , \qquad (1)$$

where a is the radius of the beam. Hereafter both real and phase spaces mean their normalized spaces. We now put $\rho(r) = f(r^2)$ and $n(x) = g(x^2)$, and further $a^2 \to a$, $x^2 \to x$ and $r^2 \to r$. Then Eq. (1) becomes,

$$\int_x^a \frac{f(y)\,dy}{\sqrt{y-x}} = g(x) \ . \qquad (2)$$

When f(y) is given, g(x) is found directly by performing the integration in Eq. (2). On the other hand, when g(x) is given, Eq. (2) becomes the Volterra's integral equation of the first kind (or a variant of the equation on Abel problem) [3, 4]. For example, when the real space density is uniform, i.e., g(x) = constant, f(x) or ρ(r) can be found by solving the integral equation of Eq. (2) ,

$$Case(1.1): \qquad f(x) \propto \frac{1}{\sqrt{a-x}} \quad or \quad \rho(r) \propto \frac{1}{\sqrt{a^2-r^2}} \ .$$

In most cases, however, we can guess their solutions by direct integration instead of dealing with the integral equation. Though already well known, two more examples are given here for convenience.

$$Case(1.2): \qquad \rho(r) \propto \delta\left(a^2-r^2\right), \quad n(x) \propto \frac{1}{\sqrt{a^2-x^2}}$$

$$Case(1.3): \qquad \rho(r) \propto 1/\left(a^2-r^2\right)^p, \quad n(x) \propto \left(a^2-x^2\right)^{1/2-p}$$

Two Dimensional Case

Since the phase space density is assumed to be uniform in the θ direction, it is written as,

$$\rho = \rho(r_1, r_2) \ , \qquad (3)$$

where r_1 and r_2 are taken as shown in Fig. 1. Then the real space density can be expressed in a similar manner as one dimensional case,

$$n(x,y) = \int \frac{2r_1 dr_1}{\sqrt{r_1^2-x^2}} \int \frac{2\rho(r_1,r_2)\, r_2 dr_2}{\sqrt{r_2^2-y^2}} \ . \qquad (4)$$

The integral domain not shown here depends on cases under study. A few cases are presented in the following.

Case (2.1): rectangular cross section with a uniform density

In this case Eq. (4) may be reduced to,

$$\int_x^a \frac{d\alpha}{\sqrt{\alpha-x}} \int_y^b \frac{f(\alpha,\beta)\,d\beta}{\sqrt{\beta-y}} = constant \ . \qquad (5)$$

The solution of f(α,β) is obtained in the same way as one dimensional case, and then ρ(x,y) is given as,

$$\rho(x,y) \propto \frac{1}{\sqrt{a^2-x^2}\sqrt{b^2-y^2}} \ . \qquad (6)$$

Case (2.2): elliptical (or circular) cross section with a non-uniform density

$$\rho(\alpha,\beta) \propto 1\Big/\left(1-\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}\right)^{p}, \quad n(x,y) \propto \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1-p}, \tag{7}$$

where $p < 1$.

Case (2.3): elliptical (or circular) cross section with a uniform density

The phase space density is given by a delta function ,

$$\rho(\alpha,\beta) \propto \delta\left(1-\alpha^2/a^2-\beta^2/b^2\right). \tag{8}$$

This density distribution is that of Kapchinskij-Vladimirskij [5].

III. Trajectory of Injection Point

In this section, the position and angle of injection point are described relative to the center of beam in the ring, so that the center of beam is assumed to be fixed.

One Dimensional Case

If the phase space density after injection is to be ρ(r), the number of injected particles per unit length of x-axis, j(x), must satisfy,

$$j(x) \propto x\cdot\rho(x)\,. \tag{9}$$

Here the r in ρ(r) can be regarded as equal to x because of the assumption of (3). The factor of x in Eq. (9) comes from the weighing factor of r in polar coordinates. When the injection point moves by dx, the number of injected particles dN becomes,

$$dN = j(x)\cdot dx\,. \tag{10}$$

Now we assume that the current of injected beam is constant. As dN is then proportional to time interval dt, we have,

$$dt \propto j(x)\cdot dx \quad or \quad dx/dt \propto 1/j(x)\,. \tag{11}$$

In the following, two examples are presented.

Case (1.1): n(x) = constant.

In this case, j(x) is given by,

$$j(x) \propto \frac{x}{\sqrt{a^2-x^2}}\,. \tag{12}$$

Then the expression of (11) leads to,

$$\frac{x^2}{a^2}+\frac{(t-T)^2}{T^2}=1\,, \quad or \quad x=a\sqrt{\frac{2t}{T}-\left(\frac{t}{T}\right)^2}\,, \tag{13}$$

where T is the injection time.

Case (1.3): $n(x) = (a^2-x^2)^{1/2-p}$

The result becomes,

$$a^2-x^2=a^2\left(1-\frac{t}{T}\right)^{\frac{1}{1-p}} \tag{14}$$

In Case (1.1), the density becomes uniform only after injection is completed, while the change of density during injection is described by,

$$n(x)=\frac{2}{\pi}\sin^{-1}\sqrt{\frac{X^2-x^2}{a^2-x^2}} \tag{15}$$

where X is the position of injection point and the region of x less than X is only occupied by particles.

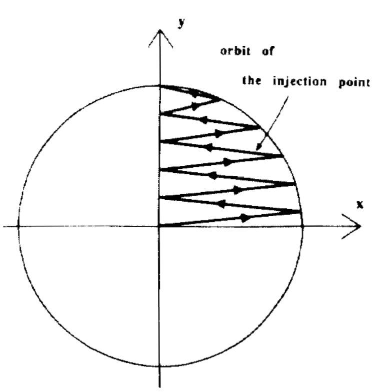

Fig. 2 Schematic Diagram for a Trajectory of Injection Point

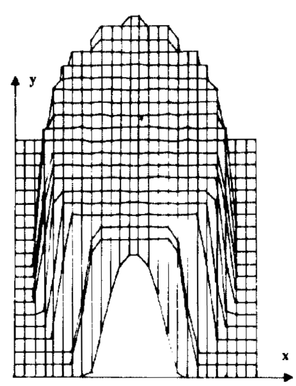

Fig. 3 Mountain View of a Two-dimensional Uniform Distribution by a simple simulation

Two Dimensional Case

To realize a certain density distribution of $\rho(r_1,r_2)$ in the phase space, the number of injected particles per unit area of real space, j(x,y), must satisfy,

$$j(x,y) \propto xy\cdot\rho(x,y) \tag{16}$$

If the phase space density is nowhere zero, the injection point has to be swept over the cross section to obtain a desired distribution. For its trajectory we can think of many patterns; for example we may inject the beam slowly increasing the position of the injection point in the radial direction while rapidly swinging it in the θ direction, or vice versa. Three examples are presented here.

Case(2.1): rectangular cross section with a uniform density

Here let the y position of the injection point increase slowly with the x position swinging rapidly. Then the y position may be regarded as constant during a cycle of the swing in the x direction. Thus the equation of the x position is reduced to that of one dimensional case, i.e., Eq.(13) with T a half period of the swing. With this solution, the equation of the y position becomes,

$$\frac{y\,dy}{\sqrt{b^2-y^2}}=(a\,function\,of\,t)\cdot dt\,. \tag{17}$$

The function of t in the equation is rapidly changing function, so that we may average it out for the motion in the y direction. Thus Eq. (17) is also reduced to one dimensional case.

Case (2.2): circular cross section with a non-uniform density

First we treat the trajectory as shown in Fig.2. The solution of the x position is found in the same way as Case (2.1),

$$x=\sqrt{a^2-y^2}\sqrt{1-(1-t/T_1)^{1/(1-p)}}\,, \tag{18}$$

where T_1 is a half period of swing in the x direction and is assumed to be independent of the y position. By averaging the motion of the x position over a period of its oscillation, the motion of injection point in the y direction becomes,

$$y=a\sqrt{1-(1-t/T)^{1/(2-p)}}\,. \tag{19}$$

It should be noted that the expression of x in Eq. (18) is valid only for a half period of T_1, but the whole pattern through injection can be easily inferred.

Second, we give here only the result in the case that the injection point is increased slowly in the radial direction, while in the θ direction it is swung rapidly,

$$\sin\theta=\sqrt{t/T_1}\,,$$
$$\frac{(1-z)^{2-p}}{2-p}-\frac{(1-z)^{1-p}}{1-p}=\frac{t/T-1}{(1-p)\,(2-p)}\,, \quad z=(r/a)^2. \tag{20}$$

Case(2.3): elliptical cross section with a uniform density

In this case j(x,y) is not zero only at the border of the beam cross section. Putting x = a·r·cosθ and y= b·r·sinθ, then j is given by,

$$j(x,y) \propto \sin\theta\cos\theta\,. \tag{21}$$

Hence the trajectory of the injection point becomes,

$$x=a\sqrt{1-\frac{t}{T}}\,, \quad y=b\sqrt{t/T}\,. \tag{22}$$

A simple numerical simulation has been made for this case, and its result is shown in Fig. 3. The blurred edge of the distribution in the figure is due to the coarse 20 × 20 meshes.

From the above cases, it can be said that the most favorable transverse distribution, namely uniform distribution, is the one to be realized most easily.

It would be worth while making an almost self-evident remark here on a relation between the injection point in the normalized space and the deflection angles of bump magnets at the injection line or in the ring. Let θ stand for the deflection angle of a magnet. Then the angles of the other magnets should be proportional to θ. Thus the position x_p and angle x'_p of injection point in the un-normalized phase space are also proportional to θ,

$$x_p \propto \theta\,,\; x'_p \propto \theta\,. \tag{23}$$

As a result, the "emittance" of injection point becomes,

$$\varepsilon=\frac{\left(x_p^2+\left(\alpha\,x_p+\beta\,x'_p\right)\right)}{\beta} \propto \theta^2 \tag{24}$$

On the other hand, the "emittance" is the square of the amplitude of the injection point on the x-axis in the normalized space. The injection point in the normalized space is, therefore, proportional to θ, and so it may be regarded as identical to θ.

IV. Longitudinal Painting

The beam is injected onto the longitudinal phase space usually with a long strip in the direction of RF phase. Therefore, only controllable may be the energy of injected or stored beams. Then we will study here about what the longitudinal distribution in the ring looks like when a beam with a simple energy distribution is injected, rather than how to inject it to get a desired distribution.

Under the assumptions in Sec. I, when a beam is injected with off momentum, zero energy spread and a distribution of n(z) in the direction of RF phase (the z direction), the phase space density becomes,

$$\rho(r) = \frac{n(z)}{2\pi r \cdot |\cos\theta|}\ , \qquad (25)$$

where $\tan\theta = \xi/z$ (see Fig. 4). In the following examples, the n(z) is assumed to be constant in the beam.

Case (3.1): right momentum with zero energy spread

The real space density $g_0(z)$ is given by,

$$g_0(z) \propto \int_z^a \frac{2dr}{\sqrt{r^2 - z^2}}\ , \qquad (26)$$

where a is a half bunch length. Hence the $g_0(z)$ becomes,

$$g_0(z) = \frac{1}{\pi a}\cosh^{-1}(a/z) \quad (z>0)\ , \qquad (27)$$

where the $g_0(z)$ is normalized as,

$$\int_{-a}^{a} g_0(z)dz = 1\ . \qquad (28)$$

At the center, the density diverges logarithmically and its derivative that is proportional to the space charge force diverges as the inverse of z. When a beam is injected with an energy spread much smaller than the RF bucket height, the distribution will look like the one in this case.

Case (3.2): off momentum with zero energy spread

The phase space density in this case is given as,

$$\rho(r) \propto \frac{1}{r\,|\cos\theta|} = \frac{1}{\sqrt{r^2 - \xi^2}}\ . \qquad (29)$$

Hence in the case of Fig. 4 the real space density becomes,

$$h_B(z) = \frac{1}{\pi a}\cosh^{-1}\sqrt{\frac{a^2}{|\xi^2 - z^2|} + y}\ , \quad \begin{cases} y = 0\ (z \geq \xi) \\ y = 1\ (\xi \geq z) \end{cases} \qquad (30)$$

with its normalization as Eq. (28). This distribution has a double-humped structure as shown in Fig. 5. They have also a singularity with logarithmic divergence. When a beam is injected into a ring with a small energy spread and with a momentum mismatch that would happen probably for a rapid cycle machine, the distribution will look like the one in this case after it becomes steady. If the space charge effect is strong, however, it will change its shape and differ from the one given here.

Case(3.3) uniform distribution on a square of phase space

The square is taken as in Fig. 6, for which the normalized density in the real space is given by,

$$g(z) = \frac{4a}{\pi}\left[\log\left|1 + 2f_1(z/a)\right| - (z/a)\log\left|f_2(z/a)\right|\right] \qquad (31)$$

$$(0 < z < a)\ ,$$

where,

$$f_1(y) = \frac{1 + \sqrt{2 - y^2}}{1 - y^2}\ , \quad f_2(y) = \frac{1 + y\sqrt{2 - y^2}}{1 - y^2}\ . \qquad (32)$$

This distribution is shown in Fig. 7 as well as the ones corresponding to uniform distributions on disks in the phase space, their radii being a and $\sqrt{2}$a, respectively. In this case, there is no singularity in the distribution.

Acknowledgment

The author would like to thank Prof. M. Kihara, the director of Accelerator Department I, for his encouragement and interest in this work.

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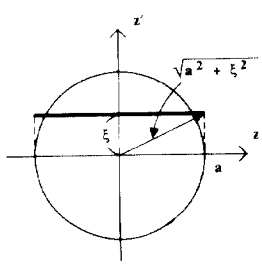

Fig. 4 A Longitudinal Distribution in the Phase Space

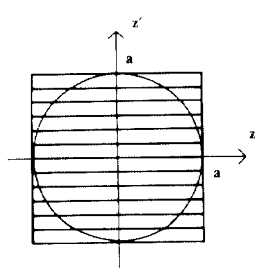

Fig. 6 Square Distribution in the Longitudinal Phase Space

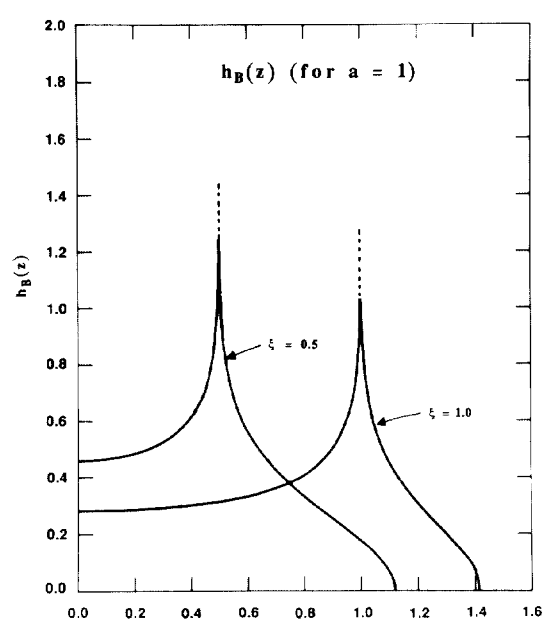

Fig. 5 Half of the Double-humped Structure of Longitudinal Distribution

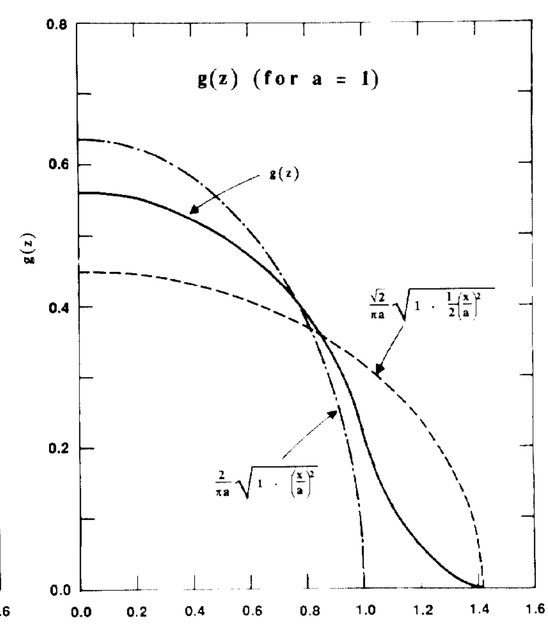

Fig. 7 Real Space Distribution for Square Phase Space Distribution