GENERATING CATALOGS OF TRANSVERSE MATCHING SOLUTIONS*

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Programs such as TRANSPORT or TRACE can find transverse beam matching solutions one at a time when given appropriate starting conditions. In the present work, an algorithm is described which rapidly finds a catalog of approximate transverse beam matching solutions. For a given initial beam, the algorithm finds the gradients of four quadrupole magnets such as to get four Twiss parameters (alpha and beta for horizontal and vertical planes) which are close to a set of desired values at the exit of a constant-energy beam line with no horizontal-vertical cross coupling and no space charge. The beam line may contain bending elements with edge corrections and other elements for which the r matrixes are known. The algorithm transforms the entrance and exit beam specifications to waist specifications, and uses the properties of waist-to-waist transport to reduce the problem from a four dimensional search to a two dimensional search.

At the Los Alamos Meson Physics Facility (LAMPF) accelerator, transverse matching is important in the low-energy transport lines (0.75 MeV), where beams from the H^+, H^-, and polarized H^- sources must be tailored for injection into the drift-tube linac; and in the transition region (100 MeV), where the beam from the drift-tube linac is injected into the side-coupled linac. Space charge has significant effects in the low-energy transport, but it is still valuable to get no-space-charge matching solutions as a starting point for solutions with space charge.

Method of Solution

The problem we wish to solve may be posed as follows: We have a section of constant-energy beam line, as shown in Fig. 1. Four quadrupoles occupy the four spaces between position 2 and 5, 6 and 9, 10 and 11, and 12 and 13. We know the r-matrixes relating the input and output trajectories, such as (for the horizontal plane)

$$\begin{pmatrix} x_o \\ x_o' \end{pmatrix} = \begin{pmatrix} r11 & r12 \\ r21 & r22 \end{pmatrix} \begin{pmatrix} x_i \\ x_i' \end{pmatrix} \qquad (1)$$

for all the remaining intervals, 1 to 2, 5 to 6, etc. Given the Twiss parameters $\alpha_{ix}, \beta_{ix}, \alpha_{iy}, \beta_{iy}$ at position 1, what are the sets of 4 quad values such that for each set, the Twiss parameters at position 14 are $\alpha_{ox}, \beta_{ox}, \alpha_{oy}, \beta_{oy}$?

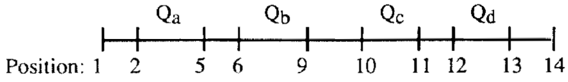

Fig. 1. A section of beam line with four quadrupoles (Q). (Positions 3, 4, 7, and 8 will be added later when we consider thin lens equivalents for Q_a and Q_b.)

Our first step is to replace the above problem by an equivalent problem involving waist-to-waist transport and replacing the first two quads by equivalent sets of thin lenses and drifts. By adding an appropriate drift at each end of the section of beam line, we may start and end at a waist (α=o). The length of drift we need to add is

$$d = \alpha/\gamma\,, \qquad (2)$$

which we need to find separately for both the horizontal and the vertical planes, and where

$$\gamma = (1 + \alpha^2)/\beta\,. \qquad (3)$$

Equation (2) applies either at the start of the section of beam line (where we use α_i and β_i), or at the end (where we use α_o and β_o). We then use the r matrix

$$R = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \qquad (4)$$

to represent the added drift from or to a waist. At the waist, we then have a β of

$$\beta_{waist} = 1/\gamma\,.$$

We now find the thin lens equivalent of a quadrupole. The elements of the r-matrix for a quadrupole are given in terms of a strength factor k and an effective length L. The factor k is related to the field gradient G in the quadrupole:

$$k = \sqrt{|K|} \qquad (5)$$

and

$$G = \beta\gamma \frac{mc}{q} K\,, \qquad (6)$$

where c is the velocity of light, m and q are the mass and charge of particles in the beam, and β and γ are relativistic parameters for the beam velocity and energy (not Twiss parameters). If the quadrupole has a field B_o at a radius a_o from the beam axis, $G=B_o/a_o$. For the plane in which the quad is focusing ($K > 0$),

$$R = \begin{pmatrix} \cos(kL) & k^{-1}\sin(kL) \\ -k\sin(kL) & \cos(kL) \end{pmatrix}, \qquad (7)$$

and for the plane in which the quad is defocusing ($K < 0$),

$$R = \begin{pmatrix} \cosh(kL) & k^{-1}\sinh(kL) \\ k\sinh(kL) & \cosh(kL) \end{pmatrix}. \qquad (8)$$

The r-matrix for a drift-thin-lens-drift combination is

$$R = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -f^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - d/f & 2d - d^2/f \\ -1/f & 1 - d/f \end{pmatrix}, \qquad (9)$$

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where d is the length of each of the drifts, and f is the focal length of the thin lens. We can make the drift-thin-lens-drift equivalent to the quad if we set

$$f^{-1} = u K L , \tag{10}$$

and compare corresponding elements in Eqs. (9) and (7) or (9) and (8). For the focusing plane (K > 0), we find u and d values of

$$u_f = \frac{\sin(kL)}{kL} ; \tag{11}$$

and $$d_f = L \frac{1 - \cos(kL)}{kL \sin(kL)} ; \tag{12}$$

and for the defocusing plane (K < 0), we find

$$u_d = \frac{\sinh(kL)}{kL} . \tag{13}$$

$$d_d = L \frac{\cosh(kL) - 1}{kL \sinh(kL)} \tag{14}$$

The d's and u's are different for the focusing and defocusing planes, and are functions of the quad strength k. However, in either plane, the d's and u's are slowly-varying functions of k, and for small k,

$$d \approx L/2 ,$$

and $$u \approx 1 .$$

We can use a procedure in which the u's are set to 1 and the d's are set to L/2 initially; and after we determine approximate values of the k's for the quads, we can find more accurate values for the d's and u's. Then we go back and find more accurate k's, etc. In the cases we have tried, this process converges in three iterations or less.

Our equivalent problem now has two sections of beam line, one of which is shown in Fig. 2, one for the horizontal plane and a slightly different one for the vertical plane. Positions 3 and 4, and positions 7 and 8 represent the entrance and exits of the thin lenses, respectively.

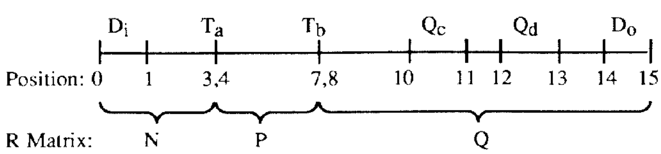

Fig. 2. Section of beam line with thin lenses T_a and T_b equivalent to Q_a and Q_b at positions 3 and 7. Drifts D_i and D_o have been added in order that the beam starts and ends with waists at positions 0 and 15.

The r matrix from position 0 to 3 is designated the N matrix and its elements are found by combining the matrixes for the initial drift D_i, the matrix for going from 1 to 2 in Fig. 1, and the matrix for the drift d (the initial drift in the drift-thin-lens-drift representing quad Q_a). Similarly matrix P represents going from 4 to 7; and matrix Q, from 8 to 15. Since we are going to search over a 2-dimensional grid of strengths for Q_c and Q_d, we assume for the current grid point that the r matrixes for Q_c and Q_d are known, and thus we have enough information to calculate the matrix Q. For the horizontal plane, we then have an r matrix

$$R = \begin{pmatrix} n_{11x} & n_{12x} \\ n_{21x} & n_{22x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f_a^{-1} & 1 \end{pmatrix} \begin{pmatrix} p_{11x} & p_{12x} \\ p_{21x} & p_{22x} \end{pmatrix} \cdot$$
$$\cdot \begin{pmatrix} 1 & 0 \\ f_b^{-1} & 1 \end{pmatrix} \begin{pmatrix} q_{11x} & q_{12x} \\ q_{21x} & q_{22x} \end{pmatrix} , \tag{15}$$

and for the vertical plane

$$R = \begin{pmatrix} n_{11y} & n_{12y} \\ n_{21y} & n_{22y} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -f_a^{-1} & 1 \end{pmatrix} \begin{pmatrix} p_{11y} & p_{12y} \\ p_{21y} & p_{22y} \end{pmatrix} \cdot$$
$$\cdot \begin{pmatrix} 1 & 0 \\ -f_b^{-1} & 1 \end{pmatrix} \begin{pmatrix} q_{11y} & q_{12y} \\ q_{21y} & q_{22y} \end{pmatrix} , \tag{16}$$

representing trajectories from the waist at position 0 to the waist at position 15.

In order to solve for the strengths f_a^{-1} and f_b^{-1} (and hence the strengths for Q_a and Q_b), we now take advantage of the fact that an r-matrix for waist-to-waist transport has a particular form.[1] That form is

$$R = \begin{pmatrix} s\, v^{-1} f^{-1} (v f^2 - \gamma_i^{-2})^{0.5} & v^{-1} f^{-1} \gamma_i^{-2} \\ -f^{-1} & s\, f^{-1} (v f^2 - \gamma_i^{-2})^{0.5} \end{pmatrix} , \tag{17}$$

where s = +1 or -1, $v = \gamma_o/\gamma_i$, and γ_o and γ_i are output and input Twiss beam parameters $\gamma = (1 + \alpha^2)/\beta$. The quantity f here relates to the focal length of the whole system from waist to waist, not to any particular quad. Equations (15) and (16) give us the r-matrix elements in terms of the quad strengths of Q_a, Q_b, Q_c, and Q_d. Equating these to the elements in (17), we then have six equations in the six unknowns f_x, f_y, K_a, K_b, K_c, and K_d:

$$r_{11x} = s_x v_x^{-1} f_x^{-1} (v_x f_x^2 - \gamma_{ix}^{-2})^{0.5} \tag{18}$$

$$r_{12x} = v_x^{-1} f_x^{-1} \gamma_{ix}^{-2} \tag{19}$$

$$r_{21x} = -f_x^{-1} \tag{20}$$

$$r_{11y} = s_y v_y^{-1} f_y^{-1} (v_y f_y^2 - \gamma_{iy}^{-2})^{0.5} \tag{21}$$

$$r_{12y} = v_y^{-1} f_y^{-1} \gamma_{iy}^{-2} \tag{22}$$

$$r_{21y} = -f_y^{-1} . \tag{23}$$

(Another pair of equations involving r_{22x} and r_{22y} are not independent of Eqs. (18) to (23), since the determinant of the r matrix is unity.) We may eliminate the f's by using Eqs. (20) and (23) to get the f's in terms of the r_{21}'s. The remaining equations become

$$r_{11x} + s_x v_x^{-1} r_{21x} (v_x r_{21x}^{-2} - \gamma_{ix}^{-2})^{0.5} = 0 \tag{24}$$

$$r_{12x} + v_x^{-1} \gamma_{ix}^{-2}\, r_{21x} = 0 \tag{25}$$

$$r_{11y} + s_y v_y^{-1} r_{21y} (v_y r_{21y}^{-2} - \gamma_{iy}^{-2})^{0.5} = 0 \tag{26}$$

$$r_{12y} + v_y^{-1}\gamma_{iy}^{-2}\, r_{21y} = 0 \qquad (27)$$

When the matrixes in Eqs. (15) and (16) are multiplied out, we find Eqs. (25) and (27) have the form

$$c_{abx}K_aK_b + c_{ax}K_a + c_{bx}K_b + c_x = 0 \qquad (28)$$

$$c_{aby}K_aK_b + c_{ay}K_a + c_{by}K_b + c_y = 0\,, \qquad (29)$$

where $c_{abx}=u_{ax}L_au_{bx}L_b(g_xn_{11x}p_{12x}q_{22x} + n_{12x}p_{12x}q_{12x})$

$c_{ax}=-u_{ax}L_a(g_xn_{11x}p_{22x}q_{22x} + g_xn_{11x}p_{12x}q_{21x} + n_{12x}p_{22x}q_{12x} + n_{12x}p_{12x}q_{11x})$

$c_{bx}=-u_{bx}L_b[(g_xn_{21x}p_{12x} + g_xn_{11x}p_{11x})q_{22x} + (n_{22x}p_{12x} + n_{12x}p_{11x})q_{12x}]$

$c_x=g_x[(n_{21x}p_{22x} + n_{11x}p_{21x})q_{22x} + (n_{21x}p_{12x} + n_{11x}p_{11x})q_{21x}] + (n_{22x}p_{22x}+n_{12x}p_{21x})q_{12x} + (n_{22x}p_{12x} + n_{12x}p_{11x})q_{11x}$

$c_{aby}=u_{ay}L_au_{by}L_b(g_yn_{11y}p_{12y}q_{22y} + n_{12y}p_{12y}q_{12y})$

$c_{ay}=u_{ay}L_a(g_yn_{11y}p_{22y}q_{22y} + g_yn_{11y}p_{12y}q_{21y} + n_{12y}p_{22y}q_{12y} + n_{12y}p_{12y}q_{11y})$

$c_{by}=u_{by}L_b[(g_yn_{21y}p_{12y} + g_yn_{11y}p_{11y})q_{22y} + (n_{22y}p_{12y} + n_{12y}p_{11y})q_{12y}]$

$c_y=g_y[(n_{21y}p_{22y} + n_{11y}p_{21y})q_{22y} + (n_{21y}p_{12y} + n_{11y}p_{11y})q_{21y}] + (n_{22y}p_{22y}+n_{12y}p_{21y})q_{12y} + (n_{22y}p_{12y} + n_{12y}p_{11y})q_{11y}$

$g_x = v_x^{-1}\gamma_{ix}^{-2}\,,\quad g_y = v_y^{-1}\gamma_{iy}^{-2}\,.$

If we now solve the horizontal plane Eq. (28) for K_a and substitute the result in the vertical plane Eq. (29), we are left with a quadratic equation for K_b:

$$K_b = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}\,, \qquad (30)$$

where $a = -c_{abx}\,c_{by} + c_{bx}\,c_{aby}\,,$
$b = -c_{ax}\,c_{by} - c_{abx}\,c_y + c_{bx}\,c_{ay} + c_x\,c_{aby}\,,$
and $c = -c_{ax}\,c_y + c_x\,c_{ay}\,.$

We can then find K_a from Eq. (28) or (29). The quantities K_a and K_b may be positive or negative (focusing or defocusing), but will not be complex for a valid solution for our 4-quad problem.

Our method for finding possible solutions is then as follows: For the current point on the grid of Q_c, Q_d gradients to be searched, we know the values of K_c and K_d for Eqs. (5) and (7)-(8). We use these and other known or approximately known values to calculate the elements of the N, P, and Q matrixes in Eqs. (15) and (16). Next we can find the values of the c's in Eqs. (28)-(29) as shown above. We now calculate two sets of values for K_a and K_b using the quadratic discussed above. We find better values for the u's and d's of Eqs. (11)-(14), put these values in the N, P, and Q matrixes, and iterate until our values of K_a and K_b converge. We find gradients corresponding to K_a and K_b from Eq. (6).

If the solution of the quadratic is complex, or the values are larger than the limits of attainable quad strengths, the sets of K_a and K_b are discarded as not being solutions. Otherwise, the second condition for waist-to-waist transport, is checked: we find if the quantity on the left side of Eq. (24) is zero or has a nearby point with the opposite sign, and similarly for Eq. (26). If this test is passed for both, the set of quad strengths is recorded as being on or near a solution.

Preliminary Implementation and Testing

One of us (P.B.) has written a computer code called MATCH4Q which implements this procedure for sections of the LAMPF beam line containing drifts, bends, bend edges, quads, and general r-matrix elements. If the K_b from Eq. (30) is complex, the code tries a nearby real value. For all sets of K values found by the code which are within the limits specified by the user, the code finds mismatch factors relating the computed Twiss parameters α and β to their target values, and can generate maps showing the regions of low mismatch factor.

We have run a number of test cases using the MATCH4Q code, and have always found at least the number of solutions that we expected. Sometimes MATCH4Q listed several sets of solution values which turned out to be different approximations to the same exact solution (as determined with the TRACE code). We also tried running a full 4-dimensional search for solutions in a few cases, using a code written just for this purpose. No solutions other than those which MATCH4Q had cataloged were found.

MATCH4Q provides for searching a grid in K_c - K_d space of up to 112 by 112 points. The time it takes to find a catalog of approximate solutions is about 3 minutes on a VAX 8700 computer for the full 112 by 112 search grid. If a grid of 70 by 70 points is used, the search time is just over one minute.

Conclusions

This method appears to be a viable way of finding approximate transverse matching solutions, which can then be refined using TRACE.[2] We have just added the procedure to the TRACE package used by the LAMPF accelerator operators. First indications are that use of the procedure will result in significant time savings when the operators are determining the best way to match the beam into one of the linacs.

References

[1] G. E. Lee-Whiting and N. Bezic, "Beam Matching with Quadrupole Lenses," Nuclear Instruments and Methods, vol. 71, pp. 61-70, 1969. See Eq. 17.

[2] Roger Cole, "TRACE User Information Manual," Los Alamos National Laboratory, LAMPF Control System Application document MP-1-3563-2, Nov. 26, 1985.