ELECTRIC POLARIZABILITY AND MAGNETIC SUSCEPTIBILITY OF SMALL HOLES IN A THIN SCREEN[1]

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I. Introduction

In the design of r.f. structures there are many applications where two or more regions are coupled through a hole in a thin metallic screen. When the hole is small compared to the r.f. wavelength, the electromagnetic properties of the hole can be represented by an induced electric dipole moment and an induced (vector) magnetic dipole moment.[2] In this paper we develop methods to obtain results for χ, the electric polarizability and $\overleftrightarrow{\psi}$, the magnetic susceptibility of a hole of general shape. A simple relation between χ and $\overleftrightarrow{\psi}$ has also been discovered.

II. Coupling Integral

Consider a cavity A of general shape, which has a small hole H on its boundary, as shown in Fig. 1.

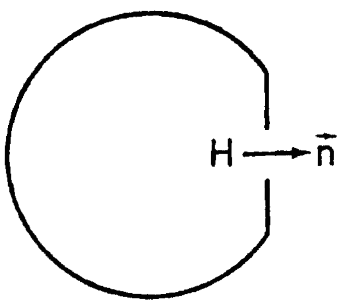

Figure 1

Following Slater's formalism[3], we write the actual field $\vec{E}(\vec{x})\, e^{j\omega t}$, $\vec{H}(\vec{x})\, e^{j\omega t}$ in the presence of the hole as an expansion in terms of the orthonormal complete set of field functions $\vec{e}_m(\vec{x})$, $\vec{h}_m(\vec{x})$ in the absence of the hole. As a result we find

$$\vec{E}(\vec{x}) = \sum_m \vec{e}_m(\vec{x}) \frac{k_m J_m}{k^2 - k_m^2} , \qquad (2.1)$$

$$Z_o \vec{H}(\vec{x}) = jk \sum_m \vec{h}_m(\vec{x}) \frac{J_m}{k^2 - k_m^2} , \qquad (2.2)$$

where $kc/2\pi = \omega/2\pi$ and $k_m c/2\pi$ are the resonant frequencies with and without the hole, and $Z_o = 120\pi$ ohms is the impedance of free space. Here J_m is a surface integral over the area of the hole

$$J_m = \int_H ds\, (\vec{n} \cdot \vec{E} \times \vec{h}_m) , \qquad (2.3)$$

where $\vec{n}$ is the outward normal at the hole.

Equation (2.1) is an integral equation since J_m depends on $\vec{E}$. Since there is a convergence problem in Eqs. (2.1) and (2.2) in the plane of the hole, we evaluate the normal component of $\vec{E}(\vec{x})$ in the vicinity of the hole using Eq. (2.1) and subsequently use this as the "far field" in treating the electrostatic problem in the immediate vicinity of the hole. Similar considerations apply to Eq. (2.2). This procedure is valid only if the dimensions of the hole are small compared to the r.f. wavelength. Therefore the normal electric field and tangential magnetic field in the vicinity of the hole (coordinates denoted by (0)) are

$$\left.\begin{aligned} E_{Az}(0) &= \sum_m e_{mz}(0) \frac{k_m J_m}{k^2 - k_m^2} , \\ Z_o \vec{H}_{At}(0) &= jk \sum_m \vec{h}_{mt}(0) \frac{J_m}{k^2 - k_m^2} . \end{aligned}\right\} \qquad (2.4)$$

Consider two general cavities, A and B, coupled by the same hole, as shown in Fig. 2.

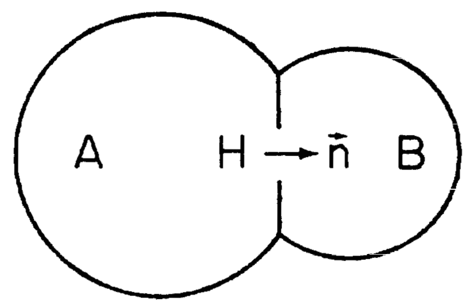

Figure 2.

The same analysis applies to cavity B and we have

$$\left.\begin{aligned} E_{Bz}(0) &= \sum_\ell e_{\ell z}(0) \frac{k_\ell J_\ell}{k^2 - k_\ell^2} , \\ Z_o \vec{H}_{Bt}(0) &= jk \sum_\ell \vec{h}_{\ell t}(0) \frac{J_\ell}{k^2 - k_\ell^2} . \end{aligned}\right\} \qquad (2.5)$$

Here the summation indices m,ℓ apply to cavity A and B respectively, and $-\vec{n}$ is the outward normal to cavity B.

In Fig. 3 we show the situation in the immediate vicinity of the hole. The surface integral in Eq. (2.3) will change if we subtract the fields $(E_{Bz} + E_{Az})/2$, $(\vec{H}_{Bt} + \vec{H}_{At})/2$ in all space, which will result in a field configuration shown in Fig. 4, where $E = (E_{Az} - E_{Bz})/2$, $\vec{H}_t = (\vec{H}_{Bt} - \vec{H}_{At})/2$. The values of J_m and J_ℓ are necessarily proportional to E and $\vec{H}_t$.

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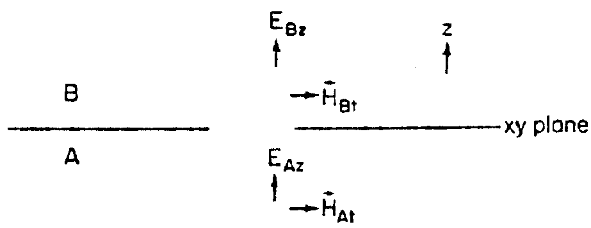

Figure 3

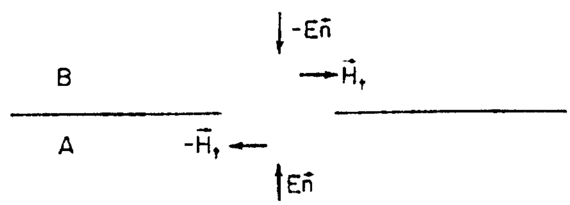

Figure 4.

III. Evaluation of the Coupling Integral

Consider the coupling integral

$$J_m = \int_H dS\, \vec{n} \cdot \vec{E}(\vec{x}) \times \vec{H}_m(\vec{x}) \qquad (3.1)$$

and expand $\vec{H}_m(\vec{x})$ in a Taylor series in $\vec{x}$ to obtain

$$J_m = J_m^0 + J_m^1 \quad . \qquad (3.2)$$

It is easy to show that an electrostatic approximation to $\vec{E}(\vec{x})$ can be used in J_m^1 to express it in terms of the electric dipole moment. In the term J_m^0, the time dependence of the fields is used to express it in terms of the magnetic dipole moment. In this way we obtain

$$J_m = k_m \chi e_{mz}(0)\, E - j\omega\mu\, \vec{H}_{mt}(0) \cdot \overleftrightarrow{\psi} \cdot \vec{H}_t \quad , \qquad (3.3)$$

in which χ and $\overleftrightarrow{\psi}$ are the electric polarizability and magnetic susceptibility of the hole. The definition of χ is

$$\chi = \frac{1}{E} \iint dxdy\, \Phi(x,y) \quad , \qquad (3.4)$$

where $\Phi(x,y)$ is the solution of the electrostatic problem with constant "far field" $\pm E\vec{n}$ as shown in Fig. 4. Similarly, $\overleftrightarrow{\psi}$ is defined by

$$\left.\begin{aligned} \psi_{xx} H_x + \psi_{xy} H_y &= \iint x\, dxdy\, H_z(x,y) \quad , \\ \psi_{yx} H_x + \psi_{yy} H_y &= \iint y\, dxdy\, H_z(x,y) \quad , \end{aligned}\right\} \qquad (3.5)$$

where $H_z(x,y)$ is the solution of the magnetostatic problem with constant "far field" $\pm\vec{H}_t$.

IV. Magnetic Susceptibility and Electric Polarizability

A. Magnetic Susceptibility

To analyze the problem with far field $H_x^H = 1$, $H_y^H = 0$, we write the scalar magnetic potential as

$$\Psi(\vec{r},z) = \pm\left[x - \iint d\vec{\sigma}\, b(\vec{\sigma})\, e^{i\vec{\sigma}\cdot\vec{r} - \sigma|z|}\right] (z \gtrless 0) \quad , \qquad (4.1)$$

where $\vec{H} = \nabla\Psi$ and $\sigma = \vec{i}k + \vec{j}\ell$, $\vec{r} = \vec{i}x + \vec{j}y$. By requiring continuity of $\Psi(\vec{r},z)$ at $z = 0$, we get an integral equation

$$\int_H d\vec{r}'\, g(\vec{r}')\, K(\vec{r},\vec{r}') = x \qquad (4.2)$$

in which

$$K(\vec{r},\vec{r}') = \frac{1}{4\pi^2} \int \frac{d\vec{\sigma}}{\sigma} e^{i\vec{\sigma}\cdot(\vec{r}-\vec{r}')} = \frac{1}{2\pi|\vec{r}-\vec{r}'|} \qquad (4.3)$$

and

$$g(\vec{r}) \equiv H_z(x,y,0) \quad . \qquad (4.4)$$

A parallel analysis for the problem of $H_x^H = 0$, $H_y^H = 1$ leads to the integral equation

$$\int_H d\vec{r}'\, h(\vec{r}')\, K(\vec{r},\vec{r}') = y \qquad (4.5)$$

in which

$$h(\vec{r}) \equiv H_z(x,y,0) \qquad (4.6)$$

The magnetic susceptibility $\overleftrightarrow{\psi}$ is related to the functions $g(\vec{r})$, $h(\vec{r})$ by the equations

$$\left.\begin{aligned} \psi_{xx} &= \int d\vec{r}\, x\, g(\vec{r}) \quad , \quad \psi_{yx} = \int d\vec{r}\, y\, g(\vec{r}) \quad , \\ \psi_{xy} &= \int d\vec{r}\, x\, h(\vec{r}) \quad , \quad \psi_{yy} = \int d\vec{r}\, y\, h(\vec{r}) \quad . \end{aligned}\right\} \qquad (4.7)$$

It is straightforward to show that

$$\psi_{xy} = \psi_{yx} \qquad (4.8)$$

which means that $\overleftrightarrow{\psi}$ can be diagonalized. If x,y are now the "diagonalized" axes, we have $\psi_{xy} = \psi_{yx} = 0$ and

$$\psi_{xx} = \int d\vec{r}\, x\, g(\vec{r}) \quad , \quad \psi_{yy} = \int d\vec{r}\, y\, h(\vec{r}) \quad . \qquad (4.9)$$

B. Electric Polarizability

We write the electrostatic potential for $E^H = 1$ as

$$\Phi(\vec{r},z) = |z| + \int d\vec{\sigma}\, e^{i\vec{\sigma}\cdot\vec{r} - \sigma|z|}\, a(\vec{\sigma}) \quad . \qquad (4.10)$$

Requiring the continuity of $E_z(x,y,0)$ within the hole gives rise to the integral equation

$$\int_H d\vec{r}'\ f(\vec{r}')\ \hat{K}(\vec{r},\vec{r}') = 1 \quad , \tag{4.11}$$

where

$$\hat{K}(\vec{r},\vec{r}') = \frac{1}{4\pi^2}\int d\vec{\sigma}\ \sigma\ e^{i\vec{\sigma}\cdot(\vec{r}-\vec{r}')} \tag{4.12}$$

and

$$f(\vec{r}) \equiv \Phi(x,y,0) \quad . \tag{4.13}$$

The electric polarizability χ is related to $f(\vec{r})$ by

$$\chi = \iint d\vec{r}\ f(\vec{r}) \quad . \tag{4.14}$$

C. Relation Between χ and $\overleftrightarrow{\psi}$

It can be shown that

$$\hat{K}(\vec{r},\vec{r}') = -\nabla^2_{\vec{r}}\ K(\vec{r},\vec{r}') = -\nabla^2_{\vec{r}'}\ K(\vec{r},\vec{r}') \tag{4.15}$$

is highly singular at $\vec{r} = \vec{r}'$. As an attempt to repair this, Eq. (4.11) is written as

$$\int d\vec{r}'\ f(\vec{r}')\ \nabla_{\vec{r}} \cdot \nabla_{\vec{r}'}\ K(\vec{r},\vec{r}') = 1 \quad , \tag{4.16}$$

which would be satisfied by

$$\int d\vec{r}'\ f(\vec{r}')\ \nabla_{r'}\ \vec{K}(\vec{r},\vec{r}') = \vec{i}\ \alpha x + \vec{j}\ \beta y \quad , \tag{4.17}$$

with the condition $\alpha + \beta = 1$. Integration of each component of Eq. (4.17) by parts and comparison with Eqs. (4.2) and (4.5) show that

$$\frac{\partial f(x,y)}{\partial x} = -\alpha g(x,y)\ ,\ \frac{\partial f(x,y)}{\partial y} = -\beta h(x,y) \quad . \tag{4.18}$$

Use of Eq. (4.18) leads to

$$\chi = \iint d\vec{r}\ f(\vec{r}) = \alpha \iint d\vec{r}\ g(\vec{r}) = \beta \iint d\vec{r}\ yh(\vec{r}), \tag{4.19}$$

from which we find the general relation between χ and $\overleftrightarrow{\psi}$

$$\chi^{-1} = \psi^{-1}_{xx} + \psi^{-1}_{yy} \quad . \tag{4.20}$$

V. Examples

Exact expressions have been obtained for the polarizability and susceptibility of an elliptical hole.[4] These can be written as

$$\frac{1}{\chi} = \frac{3}{8\pi ab}\int_o^{2\pi} d\psi \left[\frac{\cos^2\psi}{a^2} + \frac{\sin^2\psi}{b^2}\right]^{1/2} \quad , \tag{5.1}$$

$$\frac{1}{\psi_{xx}} = \frac{3}{8\pi ab}\int_o^{2\pi} d\psi\ \frac{\cos^2\psi}{a^2}\left[\frac{\cos^2\psi}{a^2} + \frac{\sin^2\psi}{b^2}\right]^{-1/2} \quad , \tag{5.2}$$

$$\frac{1}{\psi_{yy}} = \frac{3}{8\pi ab}\int_o^{2\pi} d\psi\ \frac{\sin^2\psi}{b^2}\left[\frac{\cos^2\psi}{a^2} + \frac{\sin^2\psi}{b^2}\right]^{-1/2} \quad , \tag{5.3}$$

where a and b are semi-major and semi-minor axes of the elliptical hole. The validity of Eq. (4.20) in this case is obvious. Corrsponding expressions can easily be obtained for a circular hole, where

$$\psi_{xx} = \psi_{yy} = 2\chi = 8a^3/3 \quad . \tag{5.4}$$

For small holes of general shape, variational forms for ψ_{xx} and ψ_{yy} can be used to obtain reasonably accurate susceptibilities with approximate values of $g(\vec{r})$ and $h(\vec{r})$.

VI. Summary

In this paper we showed in Section II that the fields inside a cavity with a hole can be written in terms of a coupling integral involving the tangential electric field in the plane of the hole. For a hole whose dimensions are small compared to the wavelength, this coupling integral was separated in Section III into an electric term, proportional to the (scalar) polarizability of the hole and a magnetic term proportional to the (vector) susceptibility of the hole. In Section IV we developed integral equations for the determination of these "static" geometrical parameters. These equations can be solved exactly for holes of circular and elliptical shape as shown in Section V, but only approximately for other shapes.[5] Our main result is the derivation of what appears to be a new relation between the electric polarizability and the (diagonalized) magnetic susceptibility for a hole of general shape, namely

$$\chi^{-1} = \psi^{-1}_{xx} + \psi^{-1}_{yy} \quad . \tag{6.1}$$

This can be used to test the accuracy of various numerical methods which are used to calculate these parameters.

VII. References

1. Work supported by the Department of Energy.

2. H.A. Bethe, Phys. Rev. 66, 163 (1964).

3. J.C. Slater, Microwave Electronics, Van Nostrand, 1950.

4. See, for example, R.E. Collin, Field Theory of Guided Waves, McGraw-Hill, 1960, p. 294.

5. See, for example, J. Van Bladel and C.M. Butler, Aperture Problems, Proceedings of the NATO Advanced Study Institute on Theoretical Methods for Determining the Interaction of Electomagnetic Waves with Structures, edited by J.K. Skwirzynski, Sijthoff and Noordhoff International Publishers, The Netherlands, 1981, p. 117.