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Resonance Topology

Leo Michelotti
Fermilab*, P.O.Box 500, Batavia, IL 60510

1 Separatrix: a definition.

It is hard to find a satisfactory answer to the question, "What is a resonance?" A typical response is to characterize resonances by frequency-space conditions of the form,

$$m_1\nu_1 + m_2\nu_2 + \cdots + m_p\nu_p + n = 0\ , \tag{1}$$

for integral $m_1 \ldots m_p$ and n. This definition is correct, but it ignores what should be the central feature of a resonance: its *separatrix*. The utility of a separatrix is that it globally organizes the dynamics, enabling simultaneous visualization of all the orbits and their relationships. If resonances are the building blocks of instability, then the separatrix is its mechanism. Nevertheless, establishing the concept of a separatrix for higher dimensional systems is not completely trivial. Consider, for example, Sturrock's conclusion that the first order $(1,2)$ sextupole resonance possesses unbounded orbits that pass arbitrarily close to the (phase space) origin, an error that was corrected recently by Ohnuma.[6,8] Such anomolous behavior would require that the resonance *not even possess* a separatrix.

The situation is confused further by the way that resonances appear in perturbative calculations, where they quickly become enmeshed in questions of convergence via the "small denominator" problem. This almost suggests that a resonance has more to do with the way things are calculated than with real, physical phenomena—the sort of (equally false?) feeling one sometimes gets about renormalization in quantum field theory. To offset this we emphasize that a separatrix is a topological property of a vector field. No continuous transformation of phase space, whether constructed perturbatively or inspired by God, can deform the orbits so as to make this property disappear. That is why a perturbation expansion which ignores resonances while seeking to bring a Hamiltonian into normal form must fail (globally and almost always).[1] Small denominators are not the *real* problem but only its manifestation within the context of perturbation theory. The *real* problem is that we are attempting something fundamentally impossible.

To get a better feeling for our question and for what is required of an answer, consider the following thought experiment. Suppose that you are given a one-to-one symplectic mapping, F, defined over some four-dimensional phase space and realized in an unspecified system of coordinates. (Think of F, for example, as a tracking program that models the Poincaré map of a $2\frac{1}{2}$ degree of freedom Hamiltonian system.) Starting from any number of points in phase space, you can calculate forward or backward iterates of F infinitely quickly. Further, you have unlimited capabilities for displaying these orbits on a four-dimensional graphics terminal. Given even these extraordinary tools, how would you test the simple hypothesis: "This system exhibits a first order $(1,2)$ sextupole resonance"? What *topological* features of the separatrix must be reflected in the "data" in order to confirm or deny such a statement?

There is not enough space in a short paper like this to present a full analysis of this problem. We shall short-circuit the process and simply assert what is needed to define the separatrix of an integrably resonant dynamical system on a general $2p$-dimensional phase space; a more thorough discussion is being written.[5] (In what follows, the word "orbit" refers to the set of images and preimages of a phase space point under the action of F; if P is some point in phase space, then the "orbit through P" is the set $\bigcup_{n=-\infty}^{\infty}\{F^n(P)\}$.)

ASSERTIONS:

1. At the highest level of structure, there is a way of slicing $2p$-dimensional phase space along disjoint $(p+1)$-dimensional adiabatically invariant sub-manifolds. (This may amount to little more than restating integrability, which requires that there be p invariants in involution. One of these is a Hamiltonian; the other $p-1$ label the invariant manifolds.) We shall call these slices "leaves."[2] The invariance property means that each orbit is contained within a single leaf.
2. At the next level of structure, almost all bounded orbits lie on p-tori (p-dimensional tori). (Arnold's theorem)
3. A special class of "resonant orbits" *lie on a finite set of N-periodic $(p-1)$-tori*, for some N. By saying a $(p-1)$-torus, $\mathcal{T}^{p-1}$, is N-periodic we mean that $\mathcal{T}^{p-1}$ is invariant under F^N ($F^N : \mathcal{T}^{p-1} \to \mathcal{T}^{p-1}$). Joining together the $\mathcal{T}^{p-1}$ from all the leaves produces $(2p-2)$-dimensional "tubes" of resonant orbits.
4. Each $\mathcal{T}^{p-1}$ that is unstable forms a cluster set for a set of orbits lying on zero-measure, p-dimensional manifolds. In modern terminology, they are the "alpha and omega limit sets" of these orbits, whose manifolds generalize the "stable" and "unstable" manifolds which are attached to fixed points. We shall risk abusing the terminology and call them by the same name.
5. The "separatrix" is the union of all the stable and unstable manifolds along with the periodic tori to which they are attached. It is therefore a $(2p-1)$-dimensional surface, and it partitions the $2p$-dimensional phase space, thereby serving to organize the dynamics.

The topological description of a particular resonance consists of listing the periodic tori, the $\mathcal{T}^{p-1}$, and describing how the branches of the separatrix connect them together. Testing a hypothesis, such as the one given above, consists of finding these structures in the system of interest.

Of course, knowing what to look for is not the same as knowing how to find it. In two-dimensional phase spaces, an N-periodic 0-torus is simply a fixed point of the iterated mapping F^N, and any fixed point algorithm employing Newton's method (gradient search) will usually locate it. (Of course, you must choose a good starting point and somehow specify the appropriate N, but once that is done, the algorithm converges rapidly.) In contrast to this happy situation, there is no general purpose procedure for finding higher dimensional periodic tori. The difficulty is that Hamiltonian systems are symplectic: in a sense, resonant orbits are attractors, but *the measure of their basin of attraction is zero.* Think of Newton's method as a replacement rule that substitutes a contractive mapping for a given one in such a way that an attractor of the former is a fixed point of the latter. *Does a similar rule exist for higher dimensional resonances?* We pose this as a

PROBLEM: Given a symplectic map, F, does there exist a dissipative mapping, G, constructible from F, such that attractors of G are periodic tori of F?

2 Separatrix: an example.

To illustrate all of this, we shall draw the separatrix for the first order $(1,2)$ sextupole resonance. Visualizing a four-dimensional figure like this is a little involved, but not impossible. One method is to take a sequence of three-dimensional slices, much as one would present a cube to a two-dimensional creature by slicing it from bottom to top. Of course, we must take some care in arranging the slices; our two-dimensional friend would form a distorted concept of a cube were it presented sliced along a diagonal. We shall obtain a good representation of the four-dimensional dynamics by drawing the separatrix within each three-dimensional leaf of Assertion 1 and observing its bifurcations as we pass through the leaves.

The model Hamiltonian, defined over a punctured phase space, is

$$H = \nu_1 I_1 + \nu_2 I_2 + g I_1^{1/2} I_2 \cos(\delta_1 + 2\delta_2 + n\theta + \phi)$$

I_1 and I_2 are amplitude variables conjugate to the phase variables δ_1 and δ_2; θ is the independent variable; the numbers g and ϕ are functionals of the sextupole distribution.[3] By a canonical transformation we can define new coordinates

$$\begin{aligned} J_1 &= (I_1 + 2I_2)/5 \\ J_2 &= (2I_1 - I_2)/5 \\ \xi_1 &= \delta_1 + 2\delta_2 + n\theta + \phi \\ \xi_2 &= 2\delta_1 - \delta_2 \end{aligned} \tag{2}$$

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[1] Even so, the first few low-order terms of an asymptotic series which includes resonances may contain useful information on the macroscopic structure of the flow.[2,4]

[2] An extraordinary example of dividing a space into lower dimensional manifolds can be found in A.Szulkin, "R^3 is the Union of Disjoint Circles," *American Mathematical Monthly* 90(9),640 (1983).

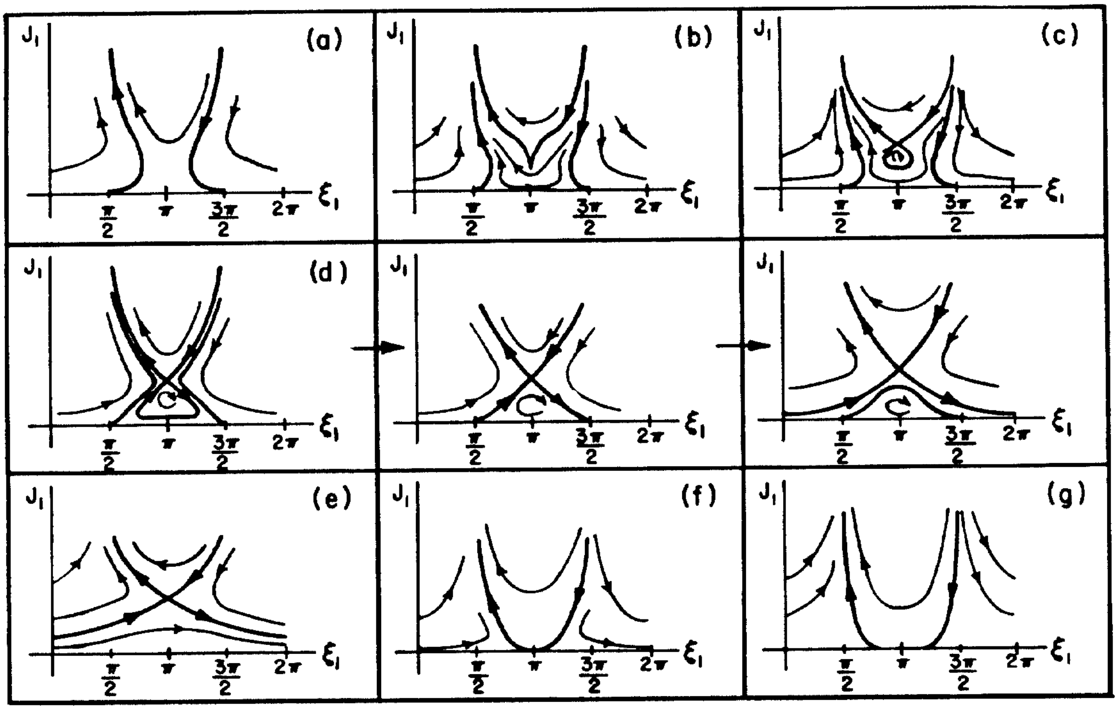

Figure 1: Projected slices of a four-dimensional $(1,2)$ separatrix.

for which the Hamiltonian function, K, is given by

$$K = J_1\Delta + J_2\Gamma + gI_1^{1/2}I_2\cos\xi_1 \tag{3}$$

where $\Delta \equiv \nu_1 + 2\nu_2 + n$ and $\Gamma \equiv 2\nu_1 - \nu_2$. It is expected that Δ is a small quantity. Indeed, for this Hamiltonian to be at all interesting Δ must be small enough so that $J_1\Delta$ is comparable in magnitude to the resonant term. ξ_2 does not appear in K, which means that (a) the invariant tori run parallel to ξ_2 and (b) J_2 is invariant and can label the leaves. The Hamiltonian flow, projected along ξ_2, is given by the vector field

$$\begin{aligned} \dot{J}_1 &= gI_1^{1/2}I_2\sin\xi_1 \\ \dot{\xi}_1 &= \Delta + gI_1^{-1/2}(\frac{1}{2}I_2 + 2I_1)\cos\xi_1 \end{aligned} \tag{4}$$

Resonant orbits of K are projected into fixed points of Eq.s(4). We shall call "regular" those resonant orbits for which $\sin\xi_1 = 0$ and "irregular" those for which either $I_1 = 0$ or $I_2 = 0$.

Symmetries of the projected flow will allow us to confine our attention to the parameter quadrant: $\Delta > 0$, $g > 0$. Clearly, if we simultaneously change the sign of both these quantities, the flow simply changes direction. Changing the sign of g alone can be compensated for by the transformation $\xi_1 \to \xi_1 + \pi$. Finally, changing the sign of Δ alone amounts to performing both previous transformations in succession.

In fact, as is characteristic of sextupole interactions, there are really no essential parameters in the problem: both Δ and g can be made to vanish by a simple scaling transformation. Let us define $\kappa \equiv \Delta/g$, and scale the amplitude variables by κ^2.

$$j_{1,2} \equiv J_{1,2}/\kappa^2 \quad i_{1,2} \equiv I_{1,2}/\kappa^2$$

Then the level sets—which determine the topology of the flow—of the function

$$\mathcal{K} \equiv g^2(K - J_2\Gamma)/\Delta^3 = j_1 + i_1^{1/2}i_2\cos\xi_1$$

are identical to those of K. Further, $\mathcal{K}$ can act as a true Hamiltonian for the scaled variables provided we simultaneously rescale $\theta \to \theta\Delta^3/g^2$.

The separatrix is sketched in Figure 1. Each frame shows its intersection with a single three-dimensional J_2 leaf projected along the ξ_2 direction onto the (ξ_1, J_1) plane. A few points should be kept in mind while scanning these pictures. First, the ξ_1 axis corresponds *not* to $J_1 = 0$ but to $J_1 = -2J_2$ ($I_1 = 0$), when $J_2 < 0$, and to $J_1 = \frac{1}{2}J_2$ ($I_2 = 0$), when $J_2 > 0$. Second, the dynamical range of ξ_1 is 6π: we are viewing only one-third of the full projection; each picture is repeated twice. Third, remember that a "fixed point" in the diagram is the projection of a period-three 1-torus, a closed curve corresponds to a 2-torus, and an open (unbounded) curve corresponds to a two-dimensional surface.

We now describe the separatrix: (a) For J_2 large and negative all orbits are unbounded except the irregular resonant orbits, which are pinned to to the surface $I_1 = 0$ at phases $\xi_1 \simeq \pm\pi/2$. (b) As J_2 increases, a local bifurcation, or catastrophe, occurs on the leaf $J_2 = -\frac{1}{30}\kappa^2$. It is heralded by the appearance of a new branch of the separatrix connected non-transversally (forming a cusp) to a 3-periodic 1-torus. (c) That torus splits, and for $-\frac{1}{30}\kappa^2 < J_2 < -\frac{1}{40}\kappa^2$ there is a single class of bounded orbits. (d) A global bifurcation, a saddle-switch, occurs on the leaf $J_2 = -\frac{1}{40}\kappa^2$. At this precise value, the surface $I_1 = 0$ is stable for phases that are 2π-equivalent to the range $\pi/2 < \xi_1 < 3\pi/2$. On the leaves $-\frac{1}{40}\kappa^2 < J_2 < 0$ there are two classes of bounded orbits. The first, say Class A, is as before and is characterized by a bounded phase, $\pi/2 < \xi_1 < 3\pi/2$. The second, Class B, has an unboundedly increasing phase ξ_1. (A better way of saying this: Class A orbits lie on 3-periodic 2-tori, while Class B orbits lie on invariant 2-tori. Or: the underlying invariant manifold of a Class A orbit is disconnected.) The entire surface $I_1 = 0$ is now locally stable. (e) For $0 < J_2 < \frac{1}{10}\kappa^2$ the Class A orbits have disappeared; Class B orbits are still bounded. (f) When $\frac{1}{10}\kappa^2 < J_2$ Class B has disappeared as well. All orbits are once more unbounded, except the two unpinned irregular resonant orbits in the plane $I_2 = 0$ which begin at $\xi_1 \simeq \pi$ at $J_2 = \frac{1}{10}\kappa^2$ and (g) wander to $\xi_1 \simeq \pm\pi/2$ as $J_2 \to \infty$.

3 Adiabatic resonance widths.

Except for the irregular resonant orbits pinned on $I_1 = 0$ and $I_2 = 0$, the $(1,2)$ resonance possesses no bounded orbits on the leaves for which $J_2 < -\frac{1}{30}\kappa^2$ or $\frac{1}{10}\kappa^2 < J_2$, whereas between these leaves bounded orbits fill some volume of phase space. This is the general behavior of all resonances, except the quadrupole resonances for which all orbits are either bounded or unbounded: the region of bounded orbits slowly shrinks as the resonance is approached. One quantitative measure of this approach to global instability is the "resonance width." Ohnuma has pointed out that this term has been used in a variety of imprecise ways by different authors.[7] Vaguely speaking, it refers to the size of the smallest strip in tune space which is centered on the resonance line, Eq.(1), and outside of which a beam is stable. This definition remains ambiguous, because it depends on the size and shape of the beam as well as on the experimental setup—e.g., on whether the resonance is approached adiabatically or the beam is suddenly injected into the resonant situation. In order to avoid beam parameters entirely, we shall associate an "adiabatic resonance width" with each individual orbit. That is, we imagine initializing an orbit in phase space with control parameters set far from resonance, then approaching the resonance very slowly, and finally noting when the orbit becomes unbounded.

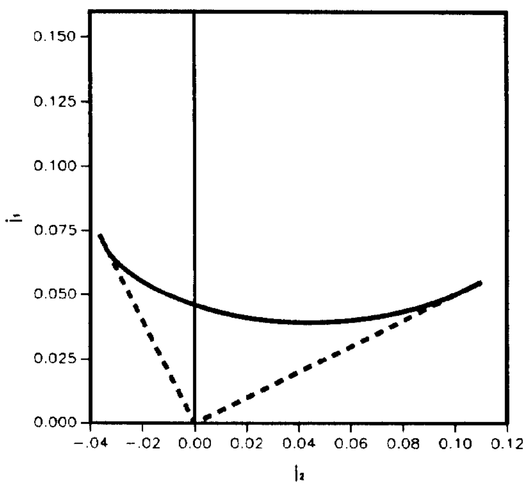

Figure 2: Resonance width master curve.

For the $(1,2)$ resonance of our example this means beginning with $\kappa \approx \infty$ and letting $\kappa \to 0$ on a time scale much greater than $\max(1/\nu_1, 1/\nu_2)$. At $\kappa = \infty$ all orbits are harmonic oscillator orbits, the variables I_1, I_2, J_1 and J_2 are conserved separately, and we can label an orbit with any two of the four initial values, I_1^{in}, I_2^{in}, J_1^{in} and J_2^{in}.[3] According to the usual adiabatic theorems the variation of an orbit as κ approaches zero will be regulated by the adiabatic invariance of the action integrals.[1] Because J_2 is a constant of motion for fixed κ, we can take $J_2 = \frac{1}{6\pi}\oint J_2 d\xi_2$ itself as the first adiabatic invariant. To the second we attach the symbol $\mathcal{A} \equiv \oint J_1 d\xi_1$, whose value is $\mathcal{A}^{in} = 6\pi J_1^{in}$.

What happens to an orbit as κ slowly decreases depends critically on the sign of J_2^{in}. For $J_2^{in} > 0$ the diagrams of Figure 1e-g are the relevant ones, and we now must think of them as *flow* diagrams for the projected Hamiltonian (see Eq.(3)) rather than *mapping* diagrams of the function F. As κ decreases the separatrix pushes downward. Each orbit remains on its leaf, $J_2 = J_2^{in}$, it maintains its value of $\mathcal{A}$, and it crosses the separatrix, thus becoming unbounded, when the area under the separatrix has decreased to $\mathcal{A}^{in}$.

For $J_2^{in} < 0$ the situation is much more interesting, as the separatrix contains two branches. Figures (1a-e) are now the relevant ones, but they must be traversed in reverse order. As κ decreases from ∞ the upper branch pushes downward, as before, but simultaneously a bubble, representing the lower branch of the separatrix, forms and begins to grow. As these two branches grow closer, approaching their merger at the saddle-switch ($\kappa^2 = -40 J_2^{in}$), orbits either are captured by the island or pass through the upper branch, depending on their values for $\mathcal{A}^{in}$. The total area under the saddle-switch is $\mathcal{A}_s = -(15 + 33\pi/4)J_2^{in}$. If $\mathcal{A}^{in} > \mathcal{A}_s$ the orbit passes through the upper branch of the separatrix; if $\mathcal{A}^{in} < \mathcal{A}_s$, then it is captured by and subsequently leaks through the lower branch. If the latter happens, $\mathcal{A}$ undergoes a discontinuous change upon passage through the separatrix, since only one of the three islands can capture the orbit. (Remember, the period 3 property refers to the phase space mapping, not the transformed flow.) As κ continues to decrease, the orbit will retain its new value for $\mathcal{A}$ as the island lifts and shrinks. Eventually—at some point before $\kappa^2 = -30 J_2^{in}$—the island becomes too small to contain the orbit.

Figure 2 contains a "master curve," drawn in the normalized (j_1^{in}, j_2^{in}) coordinates, which uses this scenario to assign resonance widths to individual orbits. The curve was computed by numerically integrating the area under the upper branch of the separatrix when $-1/40 < j_2 < 1/10$ and within the island when $-1/30 < j_2 < -1/40$. It is used in the following way. Suppose one starts an orbit at $\kappa \approx \infty$ with initial amplitude variables I_1^{in} and I_2^{in}. To find the value of κ at which the orbit becomes unbounded, first calculate J_1^{in} and J_2^{in}, using Eq.s(2), and take their ratio. The intersection of the ray $j_1^{in}/j_2^{in} = J_1^{in}/J_2^{in}$ with the "master curve" is now read off; call that point (j_1^{int}, j_2^{int}). The value of κ at which the orbit becomes unbounded is

$$\kappa = \sqrt{J_1^{in}/j_1^{int}} \ .$$

For a given resonant coupling, the adiabatic resonance width of the orbit is then determined according to $2\Delta = 2g\kappa$.

[3] Because the system is linear for $\kappa = \infty$ we can legitimately associate I_1^{in} and I_2^{in} with the initial horizontal and vertical emittances divided by 2π.[3]

A more dynamic picture is obtained by removing the $1/\kappa^2$ normalization: the curve of Figure 2 would be no longer static but sweep through the (I_1^{in}, I_2^{in}) space, converging on the origin as κ approaches zero and making orbits unbounded as it passes their initial conditions.

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