DIFFUSION OF SEPARATRIX CROSSING PARTICLES

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Adiabatic invariance theory provides a good description of trajectories in a traveling wave when the particle orbit frequency ω greatly exceeds the modulation frequency ε of the wave amplitude. However, the adiabatic invariant is destroyed by encounters with the separatrix, where ω goes to zero. Recent advances[1] in the description of this separatrix crossing process indicate that it is diffusive in energy with a characteristic diffusion rate of ε^2 per crossing. This implies a diffusion time that scales as $1/\varepsilon^3$. Numerical results which verify this scaling with ε are presented.

Adiabatic Invariance

A particle under the influence of a Hamiltonian $H(q,p,\varepsilon t)$ with slow time dependence has an adiabatic invariant $J(q,H,\varepsilon t)$ to all orders[2] in the slowness parameter ε:

$$J = I + \varepsilon J_1 + \varepsilon^2 J_2 + \dots , \tag{1}$$

where the zeroth order term is the phase space action $I(H,\varepsilon t)$. The action is directly proportional to the area enclosed by a phase space contour of constant H at a particular instant in time. In practice, the action is well conserved when the particle is far from the separatrix and $\omega >> \varepsilon$. The separatrix is the phase space contour of constant energy H_{sx}, which separates bound motion from unbound motion. Because the orbit frequency ω goes to zero on the separatrix, the action does not remain constant during a separatrix crossing.

Modulated Wave

The problem considered here is that of a charged particle under the influence of a potential wave with constant velocity and slowly modulated amplitude $A(\varepsilon t)$. In the reference frame of the wave, the dimensionless Hamiltonian is

$$H(q,p,\varepsilon t) = p^2/2 + A_0[1-A_1 \cos(\varepsilon t)] \cos(q) . \tag{2}$$

For deeply trapped particles, the orbit frequency is $\omega = A^{1/2}$, so the action is well conserved when $\varepsilon << A^{1/2}$. Because particles near the separatrix contour, which is given by $H_{sx}(\varepsilon t) = A(\varepsilon t)$, are forced to cross it repeatedly as it oscillates in phase space, there exists no invariant of the motion. A recent analysis[1] of this separatrix crossing process shows that the action is changed by a discrete amount of order ε with each crossing. If it is further assumed that successive crossings are uncorrelated, separatrix crossing particles will diffuse through phase space with a diffusion constant $D \sim \varepsilon^3$. This implies that the mean square dispersion in the energy of an ensemble of such particles has the form

$$\delta E^2(t) = \delta E_{max}{}^2 \cdot (1 - \exp(-t/\tau)) , \tag{3}$$

with a diffusion time given by $\tau \sim \varepsilon^{-3}$. This result contradicts an alternative suggestion by Menyuk[3] that the diffusion decay time should scale as $\varepsilon^{-3/2}$.

Numerical Experiments

Numerical experiments have been performed in order to test the proposed scaling law, which neglects correlations between successive separatrix crossings, and to determine the greatest modulation frequency ε for which the law remains valid. The numerical code uses parameters $A_0 = 2$ and $A_1 = .5$, so $A^{1/2} \approx 1$. The Hamiltonian equations of motion are

$$\dot{q} = p \tag{4a}$$

$$\dot{p} = (2-\cos(\varepsilon t)) \cdot \sin(q). \tag{4b}$$

The code integrates these equations over many modulation periods $2\pi/\varepsilon$, using an ensemble of 1000 particles each with initial energy $H_0 = 1.81$. Because the separatrix energy $H_{sx}(\varepsilon t)$ oscillates between the values of 1 and 3, the ensemble of particles is forced to cross the separatrix twice during each modulation period. The mean square energy dispersion $\delta E^2(t)$ of the resulting orbits is calculated after each modulation period until it saturates at the maximum value $\delta E_{max}{}^2$. Using Eq. (3), one can write

$$\ln (\delta E_{max}{}^2 - \delta E^2(t)) = \ln(\delta E_{max}{}^2) - t/\tau \tag{5}$$

so a plot of the above function vs time has a slope of $-1/\tau$, which allows one to measure the value of τ for any ε used.

Results

Experiments of the type described above have been performed for a range of modulation frequencies $.04 \leq \varepsilon \leq .7$, using a different ensemble of initial particles for runs which used the same ε. If $\tau \sim \varepsilon^{-3}$, a logarithmic plot of τ vs ε should be a straight line with slope -3. Fig. 1 shows such a plot, which convincingly verifies the scaling law for $\varepsilon < .2$.

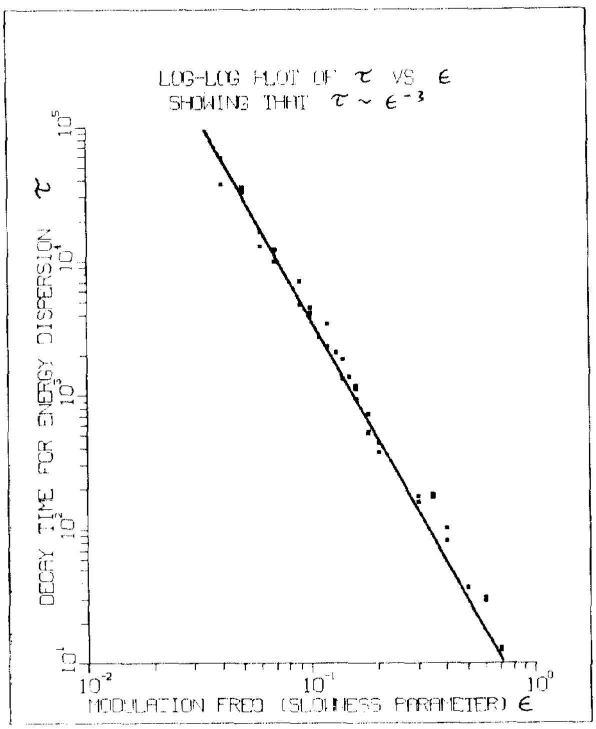

Fig. 1. Diffusion time is plotted against modulation frequency. The points are numerical data. The line has a slope of -3.

Conclusions

There exists no invariant of the motion for separatrix crossing particles. In the limit that the crossing frequency ε is small compared to the bounce frequency ω of a deeply trapped particle, the crossing process is diffusive in energy. The mean square energy dispersion of an ensemble with uniform initial energy will saturate exponentially, with decay time $\tau \sim \varepsilon^{-3}$, to some maximum value. Numerical results confirm this behavior for $\varepsilon/\omega < .2$.

References

[1] J.R. Cary, J.L. Tennyson, and D.F. Escande, "Adabatic-invariant change due to separatrix crossing," Phys. Rev. A, vol. 34, pp. 4256-75, Nov. 1986.

[2] M. Kruskal, "Asymptotic Theory of Hamiltonian and other systems with all solutions nearly periodic," J. Math Phys., vol. 3, pp. 806-28, July-Aug. 1962.

[3] C.R. Menyuk, "Particle motion in the field of a modulated wave," Phys. Rev. A, vol. 31, pp. 3282-90, May 1985.