INVARIANTS OF BETATRON MOTION AND DYNAMICAL APERTURE

J. Hagel
LEP Division, CERN,
Geneva, Switzerland

1. Summary and conclusions

In this contribution the approximate limiting stable amplitude for the non-linear betatron motion in circular accelerators containing sextupoles is derived for the purely one dimensional case of the horizontal motion. Using as input the beta functions, the sextupole positions as well as the sextupole strengths and the phase differences between the sextupole positions, the algorithm gives an estimate of the dynamical aperture at a certain observation point in the ring (see Fig. 1). The method used is to derive a generalized invariant from Hamilton's equations which leads to an integral equation for the action $J(\phi)$. This equation is solved approximately by iteration and the limit of stable motion is determined by calculating the maximum initial displacement x_0 for which the action J remains real and positive for all ϕ. The agreement between this analysis and tracking experiments in the case of a FODO-lattice with two families of sextupoles and a more general example of a LEP-type structure with insertions is good.

2. Generalized invariants

The differential equation of betatron-motion is taken from Ref. [1] and reads as :

$$x'' + Q^2 x - \frac{Q^2}{2} K'(\theta)\, \beta^{5/2}(\theta)\, x^2 = 0 \ , \qquad (1)$$

where K' and β are the sextupole strength and horizontal β-function as functions of the azimuth $\theta = \mu/Q$. This equation can be derived from the Hamiltonian :

$$H(x,p) = \frac{1}{2}(Q^2 x^2 + p^2) - \frac{Q^2}{3} K'(\theta)\, \beta^{5/2}(\theta)\, x^3, \qquad (2)$$

with Hamiltons equations :

$$dx/d\theta = \partial H/\partial p = p \quad ; \qquad dp/d\theta = -\partial H/\partial x \ . \qquad (3)$$

Multiplying the second equation (3) by p and integrating w.r.t. θ, we obtain the expression :

$$\frac{1}{2} p^2 + Q^2 \int px\, d\theta + \int px^2 f(\theta)\, d\theta = C \ , \qquad (4)$$

with $f(\theta) = -Q^2/2\, K'(\theta)\, \beta^{5/2}(\theta)$. Now we use the first equation (3) in order to eliminate the explicit θ-dependence from Eq. (4) and we find a formal (generalized) invariant, i.e. a constant of the motion depending only on the canonical variables x and p :

$$p^2 + Q^2 x^2 + 2 \int x^2 f\left(\int_{x_0}^{x} \frac{dx}{p} \right) dx = C \ . \qquad (5)$$

Finally we perform the usual transformation to action-angle variables [2] as :

$$x = J^{1/2}(\phi) \cos\phi \ , \qquad (6)$$

$$p = Q\, J^{1/2}(\phi) \sin\phi \ . \qquad (7)$$

This yields an integral equation for the unknown $J(\phi)$:

$$J(\phi) = C - \frac{2}{Q^2} F(\phi, J(\phi)) \ . \qquad (8)$$

F is the double integral contained in Eq. (5) where x and p have been replaced by Eqs. (6) and (7).

3. Iterative solution and stability limit

The method we choose to solve Eq. (8) approximately is to rewrite it as a Picard-type of iteration [3] like :

$$J_{n+1}(\phi) = C - \frac{2}{Q^2} F(\phi, J_n(\phi)) \ . \qquad (9)$$

As starting solution $J_0(\phi)$ we choose the J for the linear equation (1) which is a constant and given by :

$$J_0 = \frac{2}{Q^2} \ . \qquad (10)$$

Inserting this constant into the iteration (9) we find a first correction due to the non-linear part of Eq. (1) as :

$$J_1 = J_0 + \frac{2}{Q^2} J_0^{3/2} \int \cos^2\phi \sin\phi\, f\left(-\frac{\phi}{Q}\right) d\phi \ . \qquad (11)$$

We shall denote the integral in Eq. (11) by $G(\phi)$. From the transformations (6) and (7) and from Eq. (11) we find the retransformation from J_0 to x_0 when $p(0) = 0$:

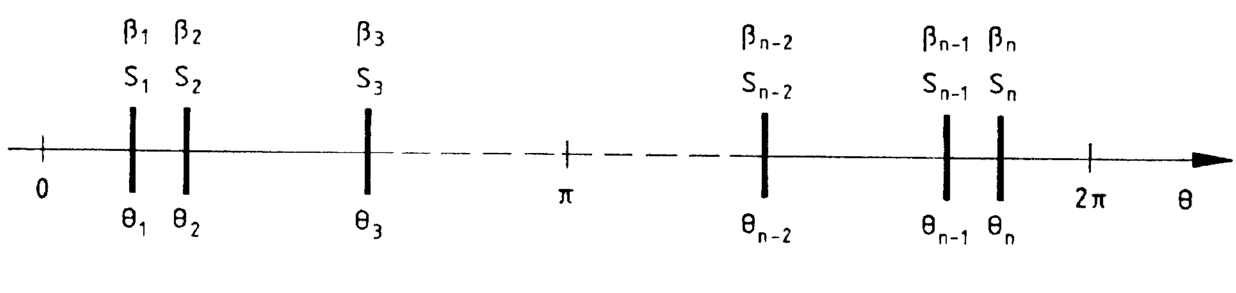

Fig.1 - General lattice layout with sextupoles S at locations θ and linear beta-functions β, $\theta = \mu/Q$. The limit is calculated at the symmetry point 0.

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$$-\frac{2}{Q^2}\,G(0)\,J_0^{3/2} + J_0 - x_0{}^2 = 0 \quad , \tag{12}$$

where we have chosen the negative sign for $J_0{}^{3/2}$.

We are now interested in the maximum stable value x_0 which, in the case of Eq. (1), leads to a bounded motion for all θ. This limit can be found by investigating the transformation (6) from x to J. We see that for real J, ϕ and bounded J this transformation must decribe a bounded motion since the cos-function is bounded. On the other hand x has to be real quantity because it represents the real motion in space. Now, if $J^{1/2}(\phi)$ becomes complex for a certain interval of ϕ, then $\cos\phi$ has to become complex and therefore behaves exponentially. Thus looking for mechanisms which render $J^{1/2}(\phi)$ complex should indicate the limit of bounded motion.

There are two possibilities for $J^{1/2}$ to become complex. The first one comes from the transformation equation (12). This equation depending on the coefficient of $J_0{}^{3/2}$ will either have two real solutions, one real double solution, or complex solutions. From Eq. (12) we may derive an exact criterion for the limiting case of one real double solution for J_0 :

$$x_0 = \frac{Q^2}{3^{3/2}\ G(0)} \quad . \tag{13}$$

Now suppose we have found a real solution for Eq. (12), then there exists a second possibility for $J^{1/2}$ to become complex (imaginary). This is obvious if we inspect Eq. (11) for $J_1(\phi)$ which can be written as :

$$J_1(\phi) = J_0\ [1 + \frac{2}{Q^2}\ J_0{}^{1/2}\ G(\phi)] \ . \tag{14}$$

Whenever for a certain interval of ϕ the expression in the brackets of Eq. (14) becomes negative, then $J_1{}^{1/2}$ becomes imaginary and thus x will be unbounded. The limit is reached when :

$$\mathrm{Min}_\phi\ G(\phi) = -\frac{Q^2}{2J_0^{1/2}} \quad , \tag{15}$$

and the limiting x_0 follows from Eq. (12).

In Fig. 2 we show $J_1(\phi)$ for two different J_0 where we used for $f(\theta) = \cos(\theta/2)$. We see a stable case (J positive) and the limiting case where the J-function just touches the ϕ-axis.

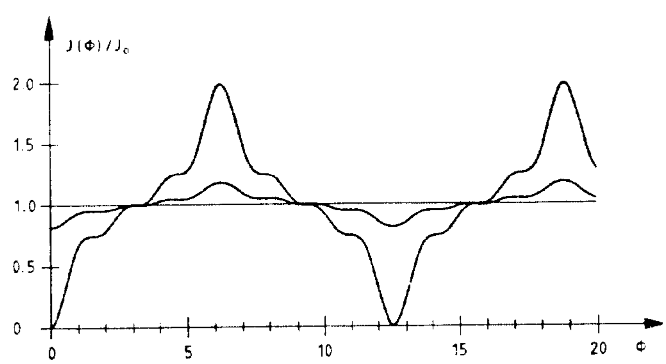

Fig.2 - Stable and limiting functions $J(\phi)$.

So at the level of the first iteration, we realize that there exist two types of transition to instability : the one generated by a complex solution for J_0 in Eq. (12) we call type 1 instability, while the second one, which is due to a negative $J_1(\phi)$ for a certain interval of ϕ, we denote by type 2 instability. Then via Eqs. (12), (13) and (15) we may calculate the two values for x_0 according to both mechanisms. The real limit then is equal to the lower result for x_0.

4. Application to circular accelerators

We may now immediately apply the theory to Eq. (1). Since the function $-\,{}^1/_2\ Q^2\ K'(\theta)\ \beta^5/_2(\theta) = f(\theta)$ is periodic in θ, we may represent it as a Fourier series. If we let $f(\theta) = f(-\theta)$, i.e. if we choose the observation point such as to be located in a symmetry centre of the magnetic structure, we may limit the series to cos-terms :

$$f(\theta) = \sum_{n=0}^{\infty} a_n \cos n\theta \ . \tag{16}$$

If we use the model of thin sextupoles, then it is possible to calculate the a_n exactly to any order [4]. Inserting $f(\theta)$ into Eq. (11) we see that $G(\phi)$ can also be found in closed form.

We checked the method for the simple case of a regular FODO-lattice with two families of sextupoles SF and SD located near the focusing and defocusing quadrupoles. We used arbitrary integrated strengths for the thin sextupoles : $(K'\ell)_F = -0.1\ m^{-2}$ and $(K'\ell)_D = 0.2\ m^{-2}$. Table 1 shows the limiting x_0 as a function of $Q = 2\pi\mu$ for values of $0.3 < Q < 0.39$. The agreement with the tracking results is very good particularly on the above third integer resonance ($Q = {}^1/_3$).

Table 1

$\mu/2\pi$	x (analytical) [cm]	x (tracking) [cm]
0.30	9.4	10.7
0.31	5.4	8.2
0.32	3.7	5.3
0.33	0.0	0.0
0.34	1.0	0.8
0.35	2.5	2.3
0.36	3.1	3.0
0.37	3.7	4.0
0.38	4.3	4.5
0.39	4.5	4.7

For the interval $0.31 < Q < 0.39$ the transition is of type 1 and a closed formula for x_0 in this Q-range is given in Ref. [4].

The tracking results have been obtained by applying a symplectic kickcode to Eq. (1) using the above example. In all cases the tracking has been extended over so many periods that the amplitude limit did not change in a certain limit of accuracy. For a detailed description see Ref. [5].

As a second example we applied our theory to a LEP-type stucture with two insertions, a phase advance of $\pi/2$ in the regular arcs and 4 families of sextupoles SD1, SF1, ... SD4, SF4. The observation point has been chosen as the low-β interaction point where $\beta_x = 1.75$ m. Fig. 3 shows the limiting x_0 [cm] at this point as a function of SF1. The line belongs to the analytical result while the dots represent the tracking.

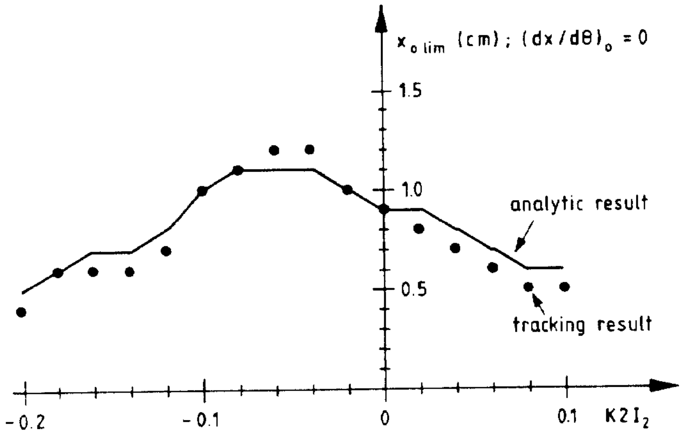

Fig.3 - Dynamical aperture and tracking results for LEP-type structure at the low-β insertions function of K21.

References

[1] H. Wiedemann, Chromaticity correction in large storage rings, PEP-Note 220 (1976).

[2] H. Goldstein, Klassische Mechanik, 7. Auflage, Akademische Verlagsgesellschaft Wiesbaden (1983).

[3] V.I. Arnold, Gewohnliche Differentialgleichungen, Springer Verlag, Berlin, Heidelberg, New York (1980).

[4] J. Hagel, Invariants of betatron motion and dynamic aperture, an analytical approach, CERN/LEP-TH/86-22 (1986).

[5] G. Guignard, J. Hagel, Sextupole correction and dynamical aperture, numerical and analytical tools, CERN/LEP-TH/85-3 (1985).