ANALYSIS OF OPTICAL PERTURBATIONS OF THE SLC ARCS*

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1. Introduction

Analytical estimation of perturbations to optical properties of a circular machine is a well-studied subject. However, the situation is different for a transport line. There are three major reasons to account for the lack of systematic studies of optical perturbations for a beam transport line. First, usually the location and geometry of a transport line is sufficiently constrainted to satisfy special requirements of ejection and injection. Therefore the lattice structure of them is often quite irregular making analytical evaluation difficult. Secondly, in the past, the transport lines are often used as an injection line to a circular machine, the optical perturbations incurred in the transport line will not translate directly into loss of luminosity or lifetime. Finally, there is some difference in the methodology of solving differential equations governing the perturbed optical functions.

For SLC, most of the reasons mentioned above do not apply anymore. First of all, the lattice design of the SLC Arc is a repetative FODO structure of 1.6 km long in each Arc. The regularity of the lattice structure makes it easier for analytical analysis. For that matter, any partial turn in the initial injection study of a big circular collider can easily be qualified as a long transport line and can be studied accordingly. On the matter of the beam quality, SLC has to rely on every pulse for collision, any deterioration of beam quality in the Arc will damage the ultimate luminosity directly. Therefore, a carefully studied and well corrected Arc system is essential for physics experiment.

About the methodology, in a circular machine, the perturbed solution of betatron function or dispersion function satisfies the periodic boundary condition in one revolution. In a transport line, the solution depends not only on the local field errors in the line, often time they also depends on the initial condition of the optical functions coming into the line. If a beam comes in mismatched, the optical parameters will be distorted whose effect overlaps with the errors generated locally to make it difficulty to separate these two effects. Therefore, as a principle, the incoming beam should be matched to the transport line lattice by tuning the unstream optics. For a matched line, the method of analysis is similar to that of a circular machine.

This paper is an attempt to establish the analytical framework in solving optical perturbations in a transport line in general and will be applied to the SLC Arc as an example. The formulation we present here should be applicable to any transport system in a straightforward way. In Section 2 the equations of motion of perturbed betatron function and dispersion function will be presented and the driving terms identified. In Section 3, the sources of possible field errors for SLC Arc system are discussed. Finally, we will estimate the magnitudes of perturbations to the optical functions for SLC Arc in Section 4.

2. Equations of Motion for Perturbed Betatron and Dispersion Functions

From the equations of motion to be shown below, any gradient error modifies the betatron function and both gradient and bending errors modify the dispersion function. If the gradient error is small we can decompose the focusing and betatron functions as follows:

$$\begin{aligned} K &= K_0 + \delta K \\ \beta &= \beta_0 + \delta\beta \end{aligned} \tag{1}$$

Since large changes in β will not be tolerated in any case, we can use an approximation that is valid for deviations $\delta\beta$ which are small relative to β itself. We define g to be the *relative* deviation, namely $g = \delta\beta/\beta$. It is shown in Courant and Snyder[1] that so long as g is small, it satisfies a differential equation which is particularly simple if we use the betatron phase $\phi = \frac{1}{\nu}\int ds/\beta$ as our independent longitudinal coordinate. Writing $\dot{g}$ for $dg/d\phi$, the equation for g after linearization is[2]

$$\ddot{g} + 4\nu^2 g = -2\nu^2\beta_0^2\delta K = f_\beta \tag{2}$$

where $\beta_0(\phi)$ is the unperturbed function, $\delta K(\phi)$ is the perturbation of the focussing function K at the longitudinal position $s(\phi)$ and ν is the number of betatron oscillation in one fundamental period of the lattice structure. It is interesting to note that the perturbed betatron function oscillates at twice the betatron frequency.

Now let us look at the case of the dispersion function,[3] consider that a particle of momentum p_0 is launched into the Arc and determines some 'central' trajectory (corresponding to a 'closed orbit' in a ring). Notice that the CT (central trajectory) is *not* necessarily the same as the design trajectory. Let's now consider the trajectories that are displaced from the CT and let x and y be the amount of the lateral displacement. If we keep only terms to first order in x, y, and $\Delta p/p_0$, the equations of motion for the transverse displacements of a nearby particle with respect to the CT are:

$$x'' = G_x(\Delta p/p_0) - K_x x - Qy \tag{3}$$
$$y'' = G_y(\Delta p/p_0) - K_y y - Qx \tag{4}$$

The curvature function $G(s)$ is proportional to the transverse field:

$$G_x = \frac{e}{p_0}B_y \quad ; \quad G_y = -\frac{e}{p_0}B_x \tag{5}$$

The focusing strength K_x and K_y are proportional to the quadrupole strength:

$$K_x = \frac{e}{p_0}\frac{\partial B_y}{\partial x} \quad ; \quad K_y = -K_x \tag{6}$$

and the coupling term Q is proportional to the skew quadrupole strength:

$$Q = \frac{e}{p_0}\frac{\partial B_y}{\partial y} \tag{7}$$

The fields and derivatives are all *to be evaluated at the CT.*

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The off-energy function is a particular trajectory for which $x = \eta_x(\Delta p/p_0)$ and $y = \eta_y(\Delta p/p_0)$ — with suitable initial conditions. Using Eq.s (3) and (4), we see that η will satisfy

$$\eta_x'' + K_x\eta_x = G_x - Q\eta_y \tag{8}$$
$$\eta_y' + K_y\eta_y = G_y - Q\eta_x \tag{9}$$

The eta-functions are 'betatron-like' oscillations driven by (a) curvature and (b) by coupling from the eta-function in the other coordinate.

Now, suppose that in any one achromat the *design* trajectory is defined by $G_x = G_0$, $G_y = 0$, $K_x = K_0$ and $Q = 0$. (Local coordinates are used.) And, with respect to this trajectory, the off-energy functions are η_{0x} and η_{0y}. Now the CT of the real machine will have different fields and derivatives and a different η. Let's define the perturbed orbit functions *with respect to the CT* by

$$G_x = G_0 + \delta\overline{G}_x \qquad G_y = \delta\overline{G}_y \tag{10}$$
$$K = K_0 + \delta\overline{K} \qquad Q = \delta\overline{Q} \tag{11}$$

They will give a perturbed off-energy function

$$\eta_x = \eta_{0x} + \overline{\eta}_x \ ; \quad \eta_y = \eta_{0y} + \overline{\eta}_y \tag{12}$$

The overline on $\delta\overline{K}$ and $\delta\overline{Q}$ is *to emphasize that they apply to values taken on the disturbed CT*. Note that we have chosen to write the *change* in η as $\overline{\eta}$ (rather than as $\delta\overline{\eta}$).

If we now insert (10),(11) and (12) into (8) and (9), we find that the anomaly $\overline{\eta}$ satisfies

$$\overline{\eta}_x'' + K_x\overline{\eta} = \delta\overline{G}_x - \eta_{0x}\ \delta\overline{K}_x - \eta_{0y}\delta\overline{Q} = f_x \tag{13}$$
$$\overline{\eta}_y'' + K_y\overline{\eta} = \delta\overline{G}_y - \eta_{0y}\ \delta\overline{K}_y - \eta_{0x}\delta\overline{Q} = f_y \tag{14}$$

with $\delta\overline{K}_x = \delta\overline{K} = -\delta\overline{K}_y$. The *perturbation* $\overline{\eta}_x$ is again a betatron-like oscillation, driven now by f_x which is a sum of three parts: a perturbed field term $\delta\overline{G}$, a perturbed gradient term $\eta_0\delta\overline{K}$ and a coupling term $\eta_0\delta\overline{Q}$, and similarly for $\overline{\eta}_y$.

If the field perturbations are known, Eqs. (13) and (14) can be solved to get the anomaly $\overline{\eta}$.

3. Field Errors Due to Trajectory Variation

There are many possible sources of field errors, but for SLC the dominant source is due to the trajectory variation, when the CT does not go through the center of a magnet. At a point whose horizontal and vertical distances from the ideal axis of an arc magnet are X and Y, the magnetic field of the Arc magnet is given by

$$\begin{aligned} B_y &= B_0 + k_0X + \frac{1}{2}S(X^2 - Y^2) \\ B_x &= k_0Y + S\ X\ Y \end{aligned} \tag{15}$$

with

$$k_0 = \frac{dB_y}{dX}, \quad S = \frac{d^2B_y}{dX^2} \tag{16}$$

are the quadrupole and setupole components. At the design energy of 50 GeV, the field values are $B_0 = 5.97$ KG, $k_0 = \pm 7.02$ KG/cm and $S = 1.63$ KG/cm^2 for focus and -2.70 KG/cm^2 for defocus magnets. If the CT passes through such a magnet at the displacement $\overline{\delta x}$, $\overline{\delta y}$ from the axis, the disturbed field functions, to *first order* in $\overline{\delta x}$ and $\overline{\delta y}$, are

$$\begin{aligned} \overline{B}_y &= B_0 + k_0\overline{\delta x} \\ \overline{B}_x &= k_0\overline{\delta y} \\ \left(\frac{\partial B_y}{\partial X}\right)_{CT} &= k_0 + S\ \overline{\delta x} \\ \left(\frac{\partial B_y}{\partial Y}\right)_{CT} &= S\ \overline{\delta y} \end{aligned} \tag{17}$$

making use of Eqs. (5) and (6), we find that

$$\begin{aligned} \delta\overline{G}_x &= K_0\ \overline{\delta x} \\ \delta G_y &= -\ K_0\overline{\delta y} \\ \delta K_x &= \mu\overline{\delta x} = -\delta K_y \\ \delta\overline{Q} &= -\mu\overline{\delta y} \end{aligned} \tag{18}$$

where $\mu = (e/p_0)S$ is the normalized sextupole strength. Here we want to remind the reader that the coordinate of a particle from magnet centerline X is made up by the distance of the particle to CT, x, and the distance from CT to magnet centerline $\overline{\delta x}$. In other words, $X = \overline{\delta x} + x$ and $Y = \overline{\delta y} + y$.

Hence the driving functions f_x and f_y of Eqs. (13) and (14) due to orbit errors $\overline{\delta x}$ and $\overline{\delta y}$ are

$$f_x = (K_0 - \eta_{0x}\ \mu)\overline{\delta x} + \eta_{0y}\ \mu\ \overline{\delta y} \tag{19}$$
$$f_y = -\ (K_0 - \eta_{0x}\ \mu)\overline{\delta y} + \eta_{0y}\ \mu\ \overline{\delta x} \tag{20}$$

In the SLC design the sextupole strengths μ of the arc magnet have been chosen to make an achromatic system for which, it turns out, the expression $(K_0 - \eta_{0x}\ \mu)$ that appears in Eqs. (19) and (20) is, when averaged over a magnet, very closely equal to zero.[4] So, for the arc magnets, the resulting driving terms for $\overline{\eta}$ reduce to

$$f_x = (\eta_{0y}\mu)\ \overline{\delta y} \tag{21}$$
$$f_y = (\eta_{0y}\mu)\ \overline{\delta x} \tag{22}$$

Our important conclusion is that alignment errors will drive errors in both η_x and η_y in any region of the arcs in which the design vertical eta, η_{0y} is not zero. Since the design of the arcs calls for rolled achromats in which there is a rather large η_{0y},[5] we must expect to find an anomalous eta driven by alignment errors.

Again, if the sources of random gradient errors are the random trajectory deviations, the driving term for the perturbed betatron function shown in Eq. (2) becomes

$$f_{\beta x} = -\ 2\ \nu_x^2\beta_{0x}^2\ \delta K_x = -2\nu^2\beta_0^2\ \mu\overline{\delta x} \tag{23}$$
$$f_{\beta y} = -\ 2\ \nu_y^2\beta_{0y}^2\ \delta K_y = \ 2\nu^2\beta_0^2\ \mu\overline{\delta x} \tag{24}$$

In the next section, we estimate the magnitude of the perturbed betatron function and anomalous eta under random alignment errors.

4. Numerical Estimates and Discussions

Suppose then that we have an impulse perturbation of strength $\delta K\Delta\phi$ located at some betatron phase ϕ_i. It will produce a small disturbance Δg with an oscillation at twice the betatron frequency and with some amplitude, say ΔA_i, namely

$$\Delta g = \Delta A_i \sin 2\nu(\phi - \phi_i); \quad (\phi > \phi_i)\ . \tag{25}$$

The initial slope $\Delta\dot{g}$ of this oscillation is just $2A_i$ which must, by Eq. (2), be equal to $-2\beta_0^2\delta K\Delta\phi$.

$$\dot{g}_i(\phi_i) = 2\nu\Delta A_i = -2\nu^2\beta_0^2\delta K\Delta\phi\ . \tag{26}$$

Consider now the value of g at the end of the arcs. Under the assumption of small errors, it will be the sum of the contribution from each magnet in the arc. Namely

$$g_L = \sum_{i=1}^{N} \Delta A_i \sin\nu(\phi_L - \phi_i) \tag{27}$$

where ϕ_L is the (undisturbed) betatron phase at the exit of the arc. Let's now assume that all focussing magnets are equivalent and give an r.m.s. contribution ΔA_F, and similarly for the defocussing magnets, for which we have an r.m.s. contribution ΔA_D.

The contributions of separate magnets combine as the squares. The contribution of each focussing magnet is

$$\langle \Delta A^2 \sin^2 \Delta\phi \rangle_F = \Delta A_F^2 \langle \sin^2 \Delta\phi \rangle_F \tag{28}$$

where $\Delta\phi$ is the phase change from each magnet to the exit. And similarly for the defocussing magnets. It turns out that the average of $\sin^2 \Delta\phi$ for both F and D magnets is just 1/2. (This average actually applies to one achromat.) Let N be the total number of magnets ($N/2$ of each type). Then the mean square g at the exit is

$$\langle g_L^2 \rangle = \frac{N}{4}(\Delta\ A_F^2 + \Delta A_D^2)\ . \tag{29}$$

If we assume the usual alignment tolerance of 100 μm, and the usual orbit correction scheme, we know that the orbit will have a random offset in the magnets of about 150 μm.[5,6] Such displacements translate to r.m.s. focusing errors of

$$\delta K_F = 1.5 \times 10^{-3}\ \text{m}^{-2}; \qquad \delta K_D = 2.4 \times 10^{-3}\ \text{m}^{-2}.$$

Furthermore, for the arcs, $N = 450$ and $\ell = 2.5$ m, using these values, we find that

$$\begin{aligned} \left(\frac{\delta\beta_{rms}}{\beta}\right)_x &= \sqrt{\langle g_L^2 \rangle}\ = 0.35 \\ \left(\frac{\delta\beta_{rms}}{\beta}\right)_y &= 0.55 \end{aligned} \tag{30}$$

Similarly, given the driving terms in Eqs. (23) and (24), the final perturbation of the dispersion function at the end of the arc have been shown to be[3]

$$(\hat{\bar{\eta}}_x)_{rms} = \sqrt{\frac{N}{2}}\ell\ \varsigma_x\ (\hat{\eta}_{0y})_{rms}\ \overline{\delta y_{rms}} \tag{31}$$

$$(\hat{\bar{\eta}}_y)_{rms} = \sqrt{\frac{N}{2}}\ell\ \varsigma_y\ (\hat{\eta}_{0y})_{rms}\ \overline{\delta x_{rms}} \tag{32}$$

where ℓ is the length of one magnet and

$$\varsigma_x = \langle \beta_x \beta_y \mu^2 \rangle_{cell}^{\frac{1}{2}} \tag{33}$$

$$\varsigma_y = \langle \beta_y^2\ \mu^2 \rangle_{cell}^{\frac{1}{2}} \tag{34}$$

Notice that ς_y is significantly different from ς_x.

From design parameters given in the SLC Design Book,[7] we have

$$\varsigma_x \approx 49\ \text{m}^{-2}, \quad \varsigma_y \approx 87\ \text{m}^{-2}$$

and

$$(\hat{\eta}_{0y})_{rms} \approx 28\ \text{mm}$$

Again, if we assume the standard alignment tolerance of 10^{-4} m, the *rms* displacements $\bar{\delta x}$ and $\bar{\delta y}$ are 1.5×10^{-4} m. We then get the following estimates for the expected anomalous eta:

$$(\hat{\bar{\eta}}_x)_{rms} \approx 8\ \text{mm}, \quad (\hat{\bar{\eta}}_y)_{rms} = 14\ \text{mm} \tag{35}$$

If expressed in terms of the percentage deviation from the design maximum etas, they are

$$(\hat{\bar{\eta}}_x)_{rms}/\hat{\eta}_{0x} = 19\% \quad \text{and} \quad (\hat{\eta})_{rms}/\hat{\eta}_{0y} = 33\% \tag{36}$$

The number we estimate here is for 100 μm random misalignment. As long as the error is less than 300 μm, the perturbation is small and linear and the analysis is still valid and the effects on optical functions should scale roughly linearly with the amount of misalignments. We only show the field errors that arise from trajectory deviations. Any other random filed or energy errors can be studied by the same formulation.

The estimates of both the perturbations of betatron functions and dispersion functions are in good agreement with the results of computer simulation done by T. Fieguth, K. Brown and R. Servranckx.[5] Methods to correct the optical perturbations are under study by the Arc group.

In the above analysis, we work out the solutions for random field errors. If the field errors are systematic, perturbed solutions satisfying proper initial conditions have been worked out in Ref. 2. The magnitudes of the pertubations have been estimated and found to be small for possible systematic errors of the Arc.

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