LANDAU DAMPING OF A MULTI-BUNCH INSTABILITY DUE TO BUNCH-TO-BUNCH TUNE SPREAD

Y.H. Chin
Deutsches Elektronen-Synchrotron DESY
Notkestraße 85, 2000 Hamburg 52, West Germany

Summary

A transverse multi-bunch instability in the presence of a split in the betatron tunes between individual bunches is studied. The formalism is summarized in a dispersion relation. It is found that the simple sinusoidal tune modulation by means of an RFQ produces no stable area in the stability diagram. Two cures are presented for expansion of the stable region. One of them, that using fractional filling of bunches, has an advantage that the width of stable band can be controlled by varying the number of missing bunches.

Introduction

In a multi-bunch ring, coherent oscillations of bunches can be coupled from bunch to bunch through structures which have impedances with long memory. A bunch leaves behind it a wake field in the structure, which perturbs the oscillation of the successive bunches. When the system makes a closed loop, all the bunches execute a coupled oscillation coherently with a certain phase difference between them.

The growth rate of the multi-bunch instability is proportional to the number of bunches, i.e., the total bunch current. In a ring with few bunches, a single bunch instability, which is attributed to a short-range wake field and depends only on the bunch's own current, is dominant and determines the intensity threshold. In a ring filled with many bunches, the multi-bunch instability gains in importance, and has an instability threshold lower than for the single bunch case.

One cure for the multi-bunch instability is to introduce a spread in the oscillation frequencies of individual bunches in order to destroy the coherence of the coupled oscillations. The spread can be provided, for example, by a modulation of the focussing force with an RF quadrupole (written as RFQ in what follows) in the transverse case, or with a subharmonic cavity in the longitudinal case.

Many existing theories assume that bunches are equally spaced, equally populated, and have the same betatron oscillation frequency. If one naively extends these analyses to the case where M bunches have different oscillation frequencies, one has M simultaneous equations. Mode frequencies are obtained by solving the M X M eigenvalue matrix, given the impedance. However, we can formulate the problem in a different way [1,2]. We rearrange the M simultaneous equations such that the eigenvalue matrix is converted to a kind of dispersion relation. If an instability is attributed to the impedance in only some of the frequencies (say, L frequencies), e.g., the frequencies of parastic cavity modes with sharp resonance peaks, the size of the matrix can be reduced to L X L. This formalism is much more useful if the number of bunches is large and the approximation of equal bunches does not hold. The purpose of this paper is to present this alternative formalism for the discussion of the stabilization of the transverse multi-bunch instability due to bunch-to-bunch betatron frequency spread. With some modifications the formalism is applicable to the longitudinal case. A bunch is assumed to oscillate rigidly, i.e., the particle distribution within a bunch does not change.

Dispersion Relation

We consider M bunches which oscillate rigidly, each bunch being represented as a single macroparticle without internal structure. Let y_l be the transverse coordinate of the l-th bunch observed at a fixed angular position θ. The l-th bunch arrives at this location at time t_l. The equation of motion for the l-th bunch is

$$\frac{d^2 y_l}{d\theta^2} + \nu_l^2 y_l = \frac{e^2 R^2}{m_o \gamma c^2} \sum_{n=-\infty}^{\infty} \sum_{j=0}^{M-1} Q_j \times W(T\frac{l-j}{M} + \tau_l - \tau_j + nT)\, y_l(\theta - 2\pi n), \qquad (1)$$

where ν_i and Q_i are the betatron tune and the total charge of the i-th bunch, e is the elementary charge, m_o is the rest mass of the particle, γ is the Lorentz factor, c is the speed of light, $T = 2\pi/\omega_o$ is the revolution time, and R is the average machine radius. The transverse wake function W(t) is defined as required by causality such that W(t) vanishes if $t < 0$. The positions of bunches deviate from equally spaced positions by a fixed time deviation τ_l. Assuming $y_l(\theta) = Y_l\, e^{-i\nu\theta}$, and introducing the impedance $Z(\omega)$, we get

$$(-\nu^2 + \nu_l^2)\, Y_l\, e^{-i\nu\theta} = \frac{e^2 R^2}{m_o \gamma c^2} \sum_{j=0}^{M-1} Q_j \frac{i}{T} \sum_{p} Z((p+\nu)\omega_o)\, Y_j\, e^{-i\nu\theta} \times e^{-2\pi i(p+\nu)(\frac{l-j}{M} + \frac{\tau_l - \tau_j}{T})}. \qquad (2)$$

If we introduce a function defined by

$$F_p = \sum_j Q_j Y_j\, e^{2\pi i(p+\nu)(\frac{j}{M} + \frac{\tau_j}{T})}, \qquad (3)$$

we can rearrange Eqs. (2) into an infinite set of equations for F_p:

$$F_q = \frac{e^2 R^2 i}{m_o \gamma c^2 T} \sum_{p=-\infty}^{\infty} \sum_{l=0}^{M-1} \frac{Q_l}{\nu_l^2 - \nu^2} Z((p+\nu)\omega_o) F_p \times e^{2\pi i(q-p)(\frac{l}{M} + \frac{\tau_l}{T})}. \qquad (4)$$

Given the impedance, the eigenvalue ν is obtained by solving the equation

$$\det[\mathbf{I} - \mathbf{M}] = 0, \qquad (5)$$

where $\mathbf{I}$ is a unit matrix, and $\mathbf{M}$ is a matrix with elements

$$M_{qp} = \frac{e^2 R^2 i}{m_o \gamma c^2 T} Z((p+\nu)\omega_o) \sum_{l=0}^{M-1} \frac{Q_l}{\nu_l^2 - \nu^2} \times e^{2\pi i(q-p)(\frac{l}{M} + \frac{\tau_l}{T})}. \qquad (6)$$

CH2387-9/87/0000-1213 $1.00 © IEEE
PAC 1987

Equation (5) is in fact a dispersion relation.

The simplest case is that the frequency $(p + \nu)\omega_0$ coincides with only one sharp resonator peak. Equation (5) then becomes

$$1 = \frac{e^2R^2i}{m_0\gamma c^2T} Z((p + \nu)\omega_0) \sum_{l=0}^{M-1} \frac{Q_l}{\nu_l^2 - \nu^2} . \qquad (7)$$

We now introduce some normalizations and rewrite Equation (7) in a more convenient form. For the moment, we assume that bunches are equally spaced, and equally populated: $Q_l = Q$. Then, the dispersion relation can be written in the normalized form

$$1 = - (U + iV) \cdot \frac{\Delta\nu}{M} \sum_{l=0}^{M-1} \frac{1}{\nu_l - \nu} \qquad (8)$$

where ν_0 is the unperturbed tune, $\Delta\nu$ is the full spread in the betatron tunes between bunches, and we have defined U and V by

$$U + iV = - i \frac{e^2R^2Q}{m_0\gamma c^2T} Z((p + \nu)\omega_0) \frac{M}{2\nu_0\Delta\nu} . \qquad (9)$$

If U and V are known and if the small ν dependence of the impedance can be neglected, M solutions are obtained for ν by solving a polynomial equation of order M given by Equation (9).

The summation appearing in Equation (8)

$$S(\nu) = \sum_{l=0}^{M-1} \frac{1}{\nu_l - \nu} \qquad (10)$$

has the singularities that result when the denominator of each term vanishes. Obviously if the eigenfrequency, ν, is further from the real axis than the distance between neighbouring singular points, the summation can be replaced by an integral to a good approximation:

$$S(\nu) \rightarrow \frac{M}{\Delta\nu} \int_{\nu_0 - \frac{\Delta\nu}{2}}^{\nu_0 + \frac{\Delta\nu}{2}} \frac{f(\nu_l)}{\nu_l - \nu} d\nu_l$$

$$= \frac{M}{\Delta\nu} \int_{-1}^{1} \frac{f(x)}{x - x_1} dx. \qquad (11)$$

Here f(x) is the normalized distribution function of tune so that $\int_{-1}^{1} f(x)dx = 1$, and

$$x_1 = \frac{\nu - \nu_0}{\Delta\nu/2} . \qquad (12)$$

The dispersion relation becomes

$$1 = - (U + iV) \int_{-1}^{1} \frac{f(x)}{x - x_1} dx , \qquad (13)$$

which is much easier to handle analytically. Roughly speaking, the distance between neighbouring singular points is $\Delta\nu/M$, so that the condition for the replacement of the summation by the integral is written as

$$\Im\nu \gg \frac{\Delta\nu}{M} . \qquad (14)$$

<u>Sinusoidal Modulation</u>

The most frequently used tune modulation pattern is a sinusoidal one generated by means of an RFQ. In this section, we concentrate on the sinusoidal modulation, and consider in detail the damping of multi-bunch instabilities due to this method.

Firstly, we again take up the case where the frequency $(p + \nu)\omega_0$ coincides with only one sharp resonator peak as in the last section. The distribution function for the sinusoidal modulation with harmonic 1 (modula M) is

$$f(x) = \frac{2}{\pi} \frac{1}{\sqrt{1 - x^2}} . \qquad (15)$$

The integration is readily carried out, with the result

$$1 = \pm (U + iV) \frac{2}{\sqrt{x_1^2 - 1}} , \quad \Re x_1 \gtrless \pm 1 \qquad (16)$$

$$= - (U + iV) \frac{2i}{\sqrt{1 - x_1^2}} , \quad |\Re x_1| < 1 \qquad (17)$$

Equation (17) states that the stability limit curve $\Im x_1 = 0$ moves on the V-axis back and forth once as x_1 varies in the range ± 1. As a result, the stability region is enclosed by the two coincident lines and has no area. This is seen more clearly from the stability diagram shown in Fig. 1. We can draw the interesting conclusion from this result that a beam cannot in principle be stabilized by means of a simple sinusoidal tune modulation, no matter how large the tune spread is, unless the impedance is purely resistive, which is rare. Of course, the growth rate can be reduced to some extent by the sinusoidal tune modulation. Thus, in the presence of other damping mechanism such as the radiation damping in an electron ring, it may be possible to reduce the growth rate of the instability to a point where the damping due to these mechanism gives a stable beam.

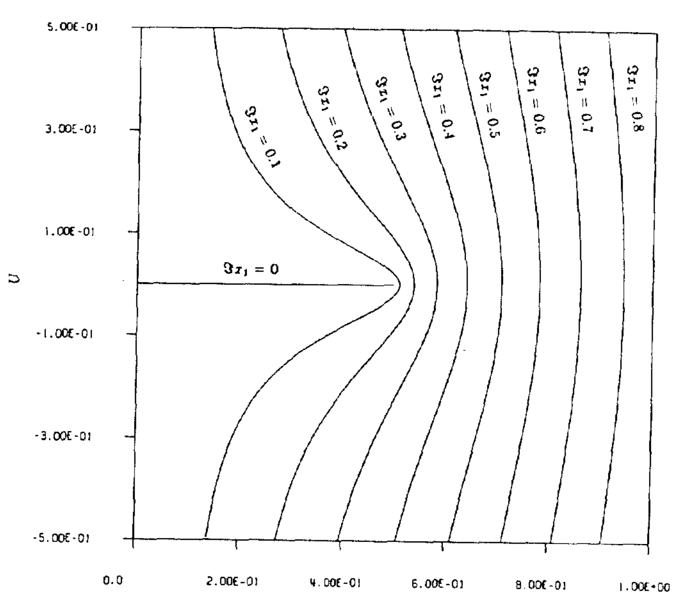

Fig. 1 The stability diagram for the sinusoidal tune modulation

We can show that the above conclusion still holds for the case where the frequency $(p + \nu)\omega_0$ coincides with two resonator peaks [2].

Let us consider various ways to obtain a stable region with non-zero area. One idea is to add higher harmonic RFQ's to the main RFQ in such a way that the modulation pattern becomes similar to a saw-tooth wave form which gives the rectangular tune distribution and the round stability diagram [2]. Another idea is to make a gap in the distribution of bunches around a ring in such a way that the rapid rise of the distribution function (15) near $x = \pm 1$, which is the cause of the coincidence of the two stability curves on the V-axis, is removed. We consider the latter idea here.

Our formalism can be readily applied to a fractionally filled beam, by pretending that a gap is still filled with bunches which consist of, say, only one particle. The number of bunches M should be counted including these pseudo bunches in the gap. As mentioned, the fact that the distribution function (15) diverges at $\theta = \arcsin x = \pm \pi/2$ causes the curve of the stability limit to be folded onto the V-axis. We therefore make gap, Δ, around $\theta = \pm \pi/2$; between $-\pi/2 - \Delta/2$ and $-\pi/2 + \Delta/2$, and between $\pi/2 - \Delta/2$ and $\pi/2 + \Delta/2$.

The stability diagram for $\Delta = \pi/6$ is plotted in Fig. 2. The stable region forms a band whose full width is approximately

$$\Delta U \simeq \frac{\Delta}{\pi} . \qquad (18)$$

To the extent that gaps are allowed, this cure is a rather promising method for enabling the sinusoidal tune modulation to work effectively.

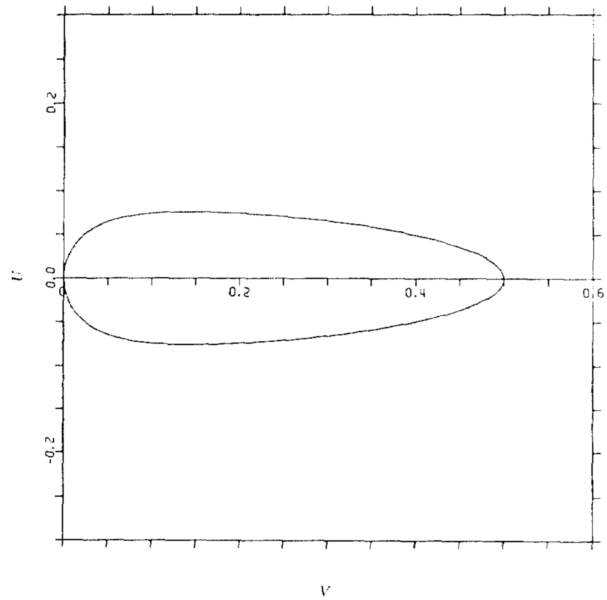

Fig. 2 The stability diagram for the fractionally filled beam. The gap parameter is $\Delta = \pi/6$.

Reference

[1] D. Möhl, "Bunch to Bunch Frequency Spread to Stabilize Coherent Oscillations in the Absence of Active Feedback", CERN Report MPS/PL/70-9 (1977).

[2] Y.H. Chin and K. Yokoya, "Landau Damping of a Multi-Bunch Instability due to Bunch-to-Bunch Tune Spread", DESY 86-097 (1986).