THERMODYNAMIC MECHANISM FOR BUNCH LENGTHENING

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A thermodynamic mechanism for bunch lengthening in electron storage rings is described, where the synchrotron radiation plays the role of a heat bath. The steady state of the bunch is determined by the minimum of free energy, in which the potential energy of bunch displacement is a key part. The result is that bunch oscillation becomes thermodynamically favored when the parasitic losses have displaced the bunch by approximately one rms width from its zero current equilibrium location. This phenomenon is independent of the synchrotron tune shifts, and is unrelated to the mechanical instabilities predicted by the linearized Vlasov equation. This threshold scaling shows that thermodynamic bunch lengthening dominates for short bunches, and a detailed calculation using the power-law impedance [1] shows that this is the probable mechanism for the bunch lengthening in SPEAR II.

Introduction

Bunch lengthening can be divided into classes depending on whether or not the particles are damped by synchrotron radiation, and whether or not the final state of the charge bunch is stationary. In the case of electron storage rings, the synchrotron radiation determines the bunch distribution. The radiation also reproduces the statistics of an ideal heat bath acting individually on the particles, and so encourages a thermodynamic analysis of the distribution. This paper is directed at the case where synchrotron radiation determines the particle distribution.

Bunch lengthening effects are produced by the longitudinal wake field. In the absence of the wake field, and assuming a constant RF gradient, the equilibrium particle distribution is Gaussian in both displacement and momentum. The wake field then produces a distortion of the potential well, and hence a distortion of the longitudinal distribution. The momentum distribution remains unchanged as long as the bunch is in thermal equilibrium, but as the bunch charge is increased, a threshold is reached where the width of the momentum distribution begins to increase also.

This phenomenon is thought to be due to instability of the equilibrium distribution, and is usually predicted by showing exponentially increasing solutions of a linearized Vlasov equation. However, when this method is applied to SPEAR II [2], the resulting prediction for the threshold current is substantially too high, compared with measurements of bunch length and momentum width [1]. When the same wake field is used in a many particle simulation, however, the result is consistent with the measurements [3]. This suggests that a different mechanism is acting to produce the instability. Such a mechanism can be found by considering bunch thermodynamics.

Thermal Distributions

The most probable distribution of an equilibrium system in contact with a heat bath is the one with the lowest free energy:

$$F = \beta E \; - \; S \; = \beta\langle E(\Gamma)\rangle \; + \; \langle \log\rho(\Gamma)\rangle \; , \tag{1}$$

where Γ is a symbol for all phase space coordinates and ρ is the probability distribution for the system to be in the state Γ. β is the inverse temperature, which is determined by the radiation statistics. The symbol $\langle\ldots\rangle$ is the phase space average: $\langle f(\Gamma)\rangle = \int d\Gamma\; \rho(\Gamma) f(\Gamma)$. S is the entropy of the charge bunch, and E is the average total energy of the charge bunch.

The condition of minimum F is the same as the condition of maximum entropy for the combined system of charge distribution plus heat bath: The quantity βE is the entropy lost by the heat bath when energy E is transferred to the charge bunch, and so F is the negative of the total entropy, plus a constant.

The condition of minimum F with the constraint that ρ remain normalized gives the variational equation $\delta(F+\lambda\langle 1\rangle)/\delta\rho = 0$, with the Boltzmann distribution as the solution:

$$\rho(\Gamma) = Z^{-1}\; e^{-\beta E(\Gamma)} \; . \tag{2}$$

Z is a normalizing constant, and the Lagrange multiplier is given by $\lambda = \log Z - 1$. This is the only stationary ρ which corresponds to an extremum of F, and it always has the Gaussian momentum distribution: The energy $E(\Gamma)$ has kinetic and potential parts, with the potential depending only on displacement:

$$E(\Gamma) = \frac{p^2}{2} \; + \; V(x) \; , \tag{3}$$

Hence, in the equilibrium case, the momentum distribution can always be factored out:

$$\rho(\Gamma) = Z^{-1}\; e^{-\beta p^2/2}\; e^{-\beta V(x)} \; . \tag{4}$$

Self-Consistent Distributions

In the case of wake fields, the interaction between any two particles is insignificant, and only the collected field of the entire charge bunch need be considered. Then one may consider the phase space distribution for a single particle in the effective potential well of the wake field induced by the distribution itself. This gives a nonlinear equation for $\rho(\Gamma)$, exactly as given by equations (2) and (3) except that now V is explicitly a function of $\rho(\Gamma)$. In the case that ρ is independent of time, the Gaussian momentum distribution factors out of the equation, leaving a self-consistent equation for $\rho(x)$ which should completely describe the bunch shape due to potential-well distortion. However, the equation for $\rho(\Gamma)$ also can admit time-dependent solutions, and these are the candidates for nonequilibrium bunch lengthening.

For the case of time dependent distributions, it is desirable to represent the single particle phase space in action-angle coordinates. The distribution function will then depend only on the action I, since the processes of filimentation and diffusion will make the density uniform along a phase space trajectory. This gives a distribution $\rho(I)$ which is constant in action- angle coordinates, even though the distribution may be time dependent in Newtonian phase space. The time dependence of the system is carried in the coordinate transformation, and the content of solving the problem is in finding that transformation.

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Thermodynamic Bunch Lengthening

The bunch lengthening problem will be treated as linear oscillation under the influence of the nonlinear wake field. This assumes constant RF gradient, and also that the synchrotron frequency is low enough so that all structure at the revolution frequency can be averaged out.

The wake field is a nonconservative force. This is due to the fact that the wake field dissipates energy into higher modes of the accelerating cavities. The higher mode loss displaces the bunch from its nominal (low-current) equilibrium position, and this effect is key to producing thermodynamic instability.

Since the charge bunch is in a quadratic confining potential, this displacement increases the potential energy of the system. This additional potential energy is what enables the thermal bunch lengthening: Normally, bunch oscillation is accompanied by an increase in the bunch energy, which makes oscillation thermodynamically unfavorable. However, if the oscillation is accompanied by lengthening of the bunch, the higher mode losses are reduced. This makes some of the potential energy of the bunch displacement available to compensate for the energy cost of bunch oscillation.

The free energy (1) can now be rewritten to show the displacement potential explicitly:

$$F = \beta E' \; - \; S \; + \; \beta \frac{1}{2} \Delta x^2 \; , \tag{5}$$

where E' is now the bunch energy calculated relative to the new (shifted) equilibrium particle trajectory. This equilibrium trajectory has been shifted by Δx from the zero-current trajectory by the higher mode losses, and the resulting potential energy appears in the third term. Note that the synchrotron radiation losses do not contribute to the problem since they do not depend on ρ, and hence the resulting potentials are constant.

Numerical Solution

The single particle longitudinal motion is determined by the Hamiltonian

$$H = \frac{p^2}{2} \; + \; \frac{x^2}{2} \; + \; U(x,t) \; , \tag{6}$$

where $U(x,t)$ is the effective wake potential

$$U(x,t) = \int dx'dp' \; u(x-x')\rho(x',p',t) \; , \tag{7}$$

and u is the indefinite integral of the wake field from a point charge: $u(x) = -\int dx \; f(x)$. The coordinates here are dimensionless, in units of the natural bunch length and momentum width. Then the Fourier transform of $f(x)$ is related to the longitudinal impedance by $\sigma_{x0}^2 \dot{V} \hat{f} = Q Z_{\|}(\omega)$, where $\sigma_{x0}, \dot{V}$, and Q are the natural bunch length in seconds, time derivative of the RF field, and bunch charge.

The nonlinear equation is solved for the distribution by iteration. The complete description of the distribution consists of the function $\rho(I)$ and the coordinate transformation $I(x,p,t)$. To expedite the calculation, several approximations are used: The coordinate transformation is calculated with the first order perturbation approximation, and, in equation (2), E is replaced by the first order action-angle Hamiltonian to represent ρ. These approximations are discussed in reference [4]. The iteration starts with a function $\rho(I)$ and a transformation $I(I_0, \Theta_0, t)$, where I_0 and Θ_0 are harmonic coordinates: $x = \sqrt{2I_0}\cos\Theta_0$ and $p = \sqrt{2I_0}\sin\Theta_0$. For the first iteration, $I = I_0$ and $\rho(I) = (2\pi)^{-1}e^{-I}$. Then the potential U is calculated from equation (7). U is represented as a Fourier series:

$$U(I_0, \Theta_0) = \sum_{m,n} U_{m,n}(I_0) \; e^{i(m\Theta_0 - n\omega t)} \; , \tag{8}$$

the new action-angle coordinates are given by

$$I = I_0 \; + \sum_{(m,n)\neq 0} \frac{m}{m - n\omega} \; U_{m,n}(I_0) \; e^{i(m\Theta_0 - n\omega t)} \; , \tag{9}$$

and the new ρ is given by

$$\rho(I) = Z^{-1} \; e^{-H(I)}, \quad H(I) = I + U_{0,0}(I) \; . \tag{10}$$

From this point, the iteration is continued until the distribution function becomes stationary.

The frequency ω is the collective oscillation frequency in units of the natural synchrotron frequency. Note that ω is a free parameter: It must be set before the iteration is done. In fact, time-dependent solutions for ρ are found for a band of frequencies ω, in addition to the static equilibrium solution. Then the free energy is calculated from equation (5), and the distribution with the smallest value of F is taken as the correct one. The smallest bunch current where the free energy of an oscillating solution is lower than that of the equilibrium solution is taken as the threshold current for bunch lengthening.

Results for SPEAR

This calculation was performed for three sets of SPEAR operating conditions, using the power-law impedance function [1]:

$$\text{Re } Z_{\|} = \begin{cases} Z_0 \; (\omega/\omega_0) & \text{for } \omega < \omega_0, \\ Z_0 \; (\omega/\omega_0)^{-a} & \text{for } \omega > \omega_0, \end{cases} \tag{11}$$

with $Z_0 = 9$ KΩ, $\omega_0 = 8.16$ ns^{-1}, and $a = 0.68$. The parameters of the different operating conditions are summarized in table 1:

Table 1: Operating Conditions for SPEAR

E_0 (GeV)	σ_{x0} (ps)	σ_{p0} (MeV)	$\dot{V}$ (PV/s)	ω_s (ms^{-1})
1.55	58.4	0.576	2.048	265
2.21	83.4	1.17	2.904	265
3.00	88.8	2.16	6.532	338

The results of the thermodynamic calculation [4] are compared with a simulation using the same impedance [3] in figures 1-3. A summary of the bunch lengthening threshold currents i_t is presented in table 2. Included are the threshold currents from measurement [1], from simulation [3], from the linearized Vlasov equation [2], and from the thermodynamic theory [4].

Scaling Laws

Dimensional analysis can be used to obtain a scaling law for bunch length versus current. Using (11) with $\omega \gg \omega_0$, obtain

$$\Delta x \propto i \; \sigma_x^{a-1} \; . \tag{12}$$

Then, minimizing F gives

$$\sigma_x \propto i^b, \quad \Delta x \propto i^b, \tag{13}$$

where $b = 1/(2-a)$. This is the same scaling law that is used in reference [1]. Hence, this result does not conflict with the scal-

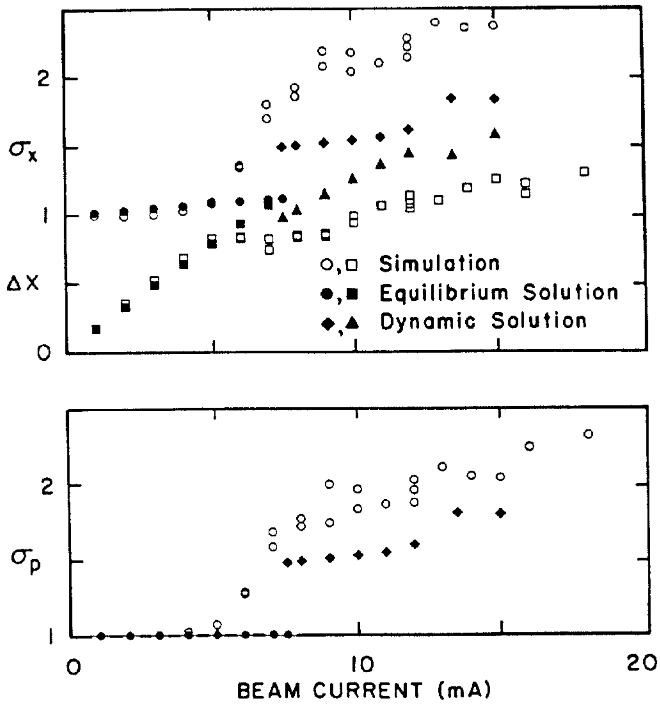

Figure 1: SPEAR Bunch Parameters at $E_0 = 1.55$ GeV

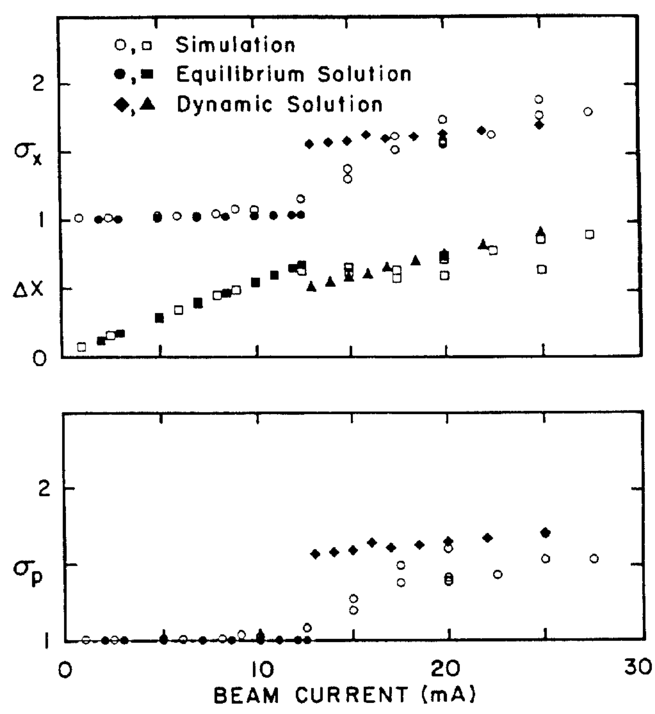

Figure 2: SPEAR Bunch Parameters at $E_0 = 2.21$ GeV

Table 2: Summary of Threshold Currents

E_0 (GeV)	i_t (mA)			
	meas.	simu.	Vlasov	therm.
1.55	3	5	13	7.5
2.21	9	12	33	13
3.00	22	30	72	28

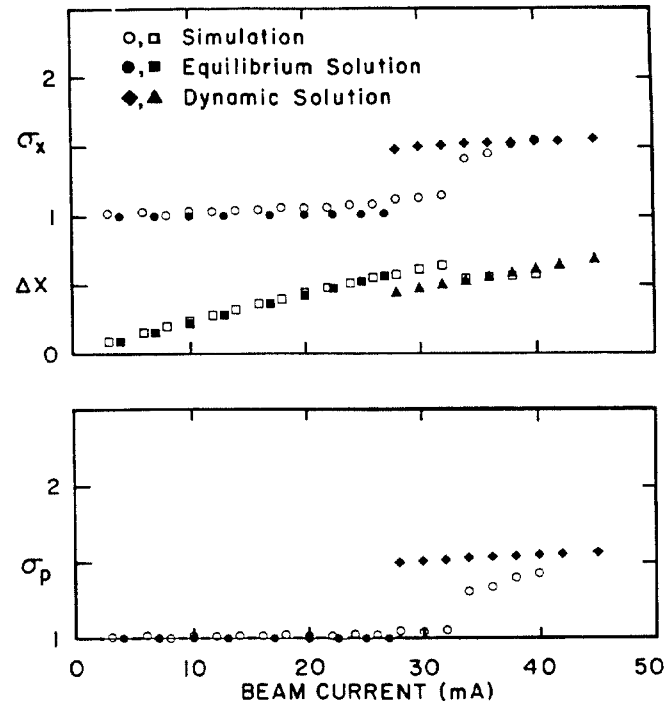

Figure 3: SPEAR Bunch Parameters at $E_0 = 3.00$ GeV

ing data shown there. The difference between the Vlasov predictions and the thermodynamic predictions presumably lies in the constant of proportionality and the behavior near threshold. The scaling argument shown above indicates that thermal bunch lengthening will become active when the higher mode losses have displaced the bunch by the order of one bunch length, and as the current increases, the bunch will lengthen so that the displacement remains tied to the bunch length.

Conclusions

The results above suggest a new mechanism for bunch lengthening in electron storage rings. This mechanism depends mainly on the higher mode losses, and their relative effect on the scale of the bunch length. It is insensitive to the detailed dynamics which produce the Vlasov instabilities which are usually called upon to explain bunch lengthening.

The calculations for SPEAR II indicate that the thermal mechanism gives a good explanation of the observations in this machine, in contrast to mechanisms based on the Vlasov equation. The general dimensional analysis predicts that the thermal mechanism should be the dominant form of bunch lengthening in very short bunch machines.

References

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