SUCCESSIVE LINEARIZATION METHOD - COMPARISON WITH AN EXACT RESULT

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Abstract

The successive linearization method has been developed to determine the dynamic aperture of storage rings. In the present paper the method is applied to a simple system that can also be solved exactly. It turns out that the dynamic aperture obtained from the successive linearization method disagrees with the exact result. This indicates that the method should be used with some caution.

1. Introduction

One of the most important quantities in beam dynamics of a storage ring is the dynamic aperture. It is defined as the boundary to that area of phase space in which particles remain stable for a given time and do not get lost.

To determine the dynamic aperture one can apply particle tracking methods[1]. However, to calculate the actual trajectories of protons and heavier particles over sufficiently many turns is often beyond the scope of present computers. Therefore the stability of particle trajectories can only be predicted for short times.

A quite different approach is taken by the successive linearization method[2]. Here it is assumed that, even for long times, the actual solutions of the particle trajectories can be sufficiently well approximated by solutions obtained from a successive linearization of the underlying differential equation. With this and some further assumptions criteria are developed which allow one to directly discern between stable and unstable trajectories and to calculate the dynamic aperture.

Of course the question arises how realistic these criteria are in view of the assumptions involved. It is therefore important to compare the successive linearization method with an exact solution as will be done in this paper. The comparison is carried through for the differential equation

$$d^2/ds^2\ x + k\ x - m \cdot x^2/2 = 0$$

with

$$d/ds\ k = d/ds\ m = m\ 0$$

Details are given in section 2. It will turn out that the successive linearization method leads to an incorrect result for the dynamic aperture even if it is driven to arbitrarily high order. Consequently some care should be taken when determining the dynamic aperture by the successive linearization method.

2. Calculations and Results

The successive linearization method has been applied to equations of the type[2]

$$d^2/ds^2\ x + k \cdot x - m \cdot x^2/2 = 0 \tag{1}$$

Here x is the displacement, $k \cdot x$ describes the qudrupole focussing whereas $m \cdot x^2/2$ is due to sextupole contributions and is assumed to be small. Eq. (1) can be obtained from the hamiltonian

$$H = p^2/2 + k/2 \cdot x^2 - m \cdot x^3/6 \tag{2}$$

An exact solution is possible if k and m do not depend on s and only this case will be investigated in the following. Then

$$x = p = \pm\ (2E - k \cdot x^2 + m\, x^3/3)^{1/2}$$

or

$$\pm \quad dx(2E - kx^2 + mx^3/3)^{-1/2} = s \tag{3}$$

E is determined from the initial conditions x_o and p_o:

$$E = p_o{}^2/2 + k/2 \cdot x_o{}^2 - m\, x_o{}^3/6 \tag{5}$$

$$E < E_c = 2 \cdot k^3/(3\, m^2)$$

where E_c is the energy below which x is bounded.

Applying next the successive linearization method to eq. (1) the first linearization consists in neglecting the term $m \cdot x^2/2$ completely which leads to

$$d^2x^{(o)}/ds^2 + k\, x^{(o)} = 0$$

with the solution

$$x^{(o)} = b \cos(\sqrt{k}\, s + \alpha)$$

b and α are determined by the initial conditions. Denoting the true dynamic aperture by A and that obtained from the successive linearization method in n^{th} order by A_n one finds

$$A_1 = \infty \tag{6}$$

In second order the deviations $u^{(o)}$ due to the term $m \cdot x^2/2$ are taken into account linearly. This leads to the equation (only the homogeneous equation needs to be discussed)

$$d^2u^{(o)}/ds^2 + [k + mb \cos(\sqrt{ka}\, s + \alpha)]u^{(o)} = 0 \tag{7}$$

With the transformations

$$z = (\sqrt{k}\, s + \alpha)/2$$

one obtains

$$d^2u^{(o)}/dz^2 + [\,4 + 4\, mb/k \cdot \cos(2z)] \cdot u^{(o)} = 0 \tag{8}$$

This is Mathieu's equation having non bounded solutions in the present case for any not too big m[3]. Thus in second order

$$A_2 = 0 \tag{9}$$

In n^{th} order one obtains n-1 corrections u to $x^{(o)}$ and the dynamic aperture must not contain area of phase space where one or several corrections u are unbounded. As a consequence A_{i+1} must be contained in A_i or

$$A_{i+1} \subset A_i \tag{10}$$

Here it amounts to (cf. eq. (7)):

$$A_i = 0 \qquad i \geq 2 \tag{11}$$

3. Conclusion

Applying the successive linearization method to eq. (1) with konstant k and m leads to the following prediction for the dynamic aperture:

It is infinite if one stops after the first linearization it is zero in any further linearization step. This is in disagreement with the exact result where the dynamic aperture is finite and determined by eq. (5). This shows that one has to be cautious when applying the successive linearization method to estimate the dynamic aperture in storage rings.

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References

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