CHARACTERISTICS OF THE BROAD RESONANCE IN THE LONGITUDINAL COUPLING IMPEDANCE*

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Introduction

Analytic calculations of the longitudinal coupling impedance of a cavity in a beam pipe have been confined to pillbox geometries[1,2,3]. These calculations require truncation and inversion of an infinite matrix corresponding to a longitudinal Fourier mode decomposition in the pillbox. More recently, computer programs have been developed which permit calculation of the longitudinal impedance for an azimuthally symmetrical obstacle in a beam pipe[4,5,6].

In this paper we use one of these programs[5] to explore the behavior of the broad resonance which occurs near the cutoff frequency of the beam pipe. In the process we reconstruct the analysis of Henke[1] for a small pillbox and determine the analytic characteristics of this broad maximum. Finally, we repeat the analysis for two small pillboxes to explore the parametrization of the interference between the two obstacles.

Analysis for a Small Pillbox

Henke's analysis consists of writing the field within the beam pipe as the sum of a source field E_z^S, E_r^S, H_ϕ^S, and a source free field E_z^I, E_r^I, H_ϕ^I which does not yet satisfy the boundary condition at the surface of the beam pipe of radius a, but which is well behaved on the axis. The relevant source fields for a relativistic beam ($\beta = 1$, $\gamma >> 1$) are

$$E_z^S \cong 0 \quad , \quad Z_o H_\phi^S = \frac{qZ_o}{2\pi r} e^{-jkz} \tag{1}$$

where q is the charge, $Z_o = 120\pi$ ohms is the impedance of free space and exp(jkct) is the time dependence of all fields. The source free field is written as

$$E_z^I = \int_{-\infty}^{\infty} dp\ F(p) \frac{J_o(Kr)}{J_o(Ka)} e^{-jpz},$$
$$Z_o H_\phi^I = -jka \int_{-\infty}^{\infty} dp\ F(p) \frac{J_o'(Kr)}{KaJ_o(Ka)} e^{-jpz} \tag{2}$$

where $K^2 = k^2 - p^2$.

The fields in the cavity can be considered independent of z for a pillbox whose length is small compared to its radius. Specifically, we can write in the pillbox

$$E_z^{II} = M \frac{P(kr)}{P(ka)} \quad , \quad Z_o\ H_\phi^{II} = -jM \frac{P'(kr)}{P(ka)} \ , \tag{3}$$

where $P(x) = Y_o(x)\ J_o(kb) - J_o(x)\ Y_o(kb)$ is the linear combination of $J_o(x)$ and $Y_o(x)$ which vanishes at x = kb, where b is the pillbox radius.

We now match E_z at r = a in Eq. (2) for all z with its value M in the pillbox at r = a, for $-g < z < g$, where 2g is the (small) axial extent of the pillbox. The inverse Fourier transform to Eq. (2) gives

$$F(p) = \frac{M}{2\pi} \int_{-g}^{g} dz\ e^{jpz} = \frac{M}{\pi p} \sin pg. \tag{4}$$

The source free magnetic field is then

$$Z_o H_\phi^I = -\frac{kaM}{2\pi} \int_{-\infty}^{\infty} \frac{dp}{p} H(p) \left[e^{+jp(g-z)} - e^{-jp(z+g)}\right] \tag{5}$$

where

$$H(p) = \frac{J_o'(x)}{xJ_o(x)} \quad , \quad x = Ka = (k^2a^2 - p^2a^2)^{1/2}. \tag{6}$$

The analytic properties of $J_o(x)$ in the complex plane permit us to write H(p) as a sum over the zeros of J(x); specifically

$$H(p) = \frac{J_o'(x)}{xJ_o(x)} = 2 \sum_{s=1}^{\infty} \frac{1}{x^2 - j_s^2} = -2 \sum_{s=1}^{\infty} \frac{1}{p^2a^2 + \beta_s^2}, \tag{7}$$

where

$$\beta_s^2 = j_s^2 - k^2a^2 = -b_s^2 \quad , \quad b_s^2 = k^2a^2 - j_s^2 \ . \tag{8}$$

The poles in Eq. (7) must be carefully considered as we close the contour in Eq. (5) to evaluate $Z_oH_\phi^I$, prior to taking the limit $g \to 0$. In fact, for $-g < z < g$ we will be closing the contour for the first term in Eq. (5) in the upper half plane, using the pole at $pa = j\beta_s$ and for the second term in the lower half plane, using theh pole at $pa = -j\beta_s$. The pole at p = 0 in each term will arbitrarily be placed in the upper half plane. For the poles at $pa = \pm b_s$, when $b_s^2 > 0$, we will use the results for $\beta_s^2 > 0$ with $\beta_s = jb_s$, corresponding to the outgoing boundary condition for the disturbance generated by the pillbox.

Matching Z_oH_ϕ in the opening of the pillbox then leads, for small g, to

$$\frac{1}{M} = \frac{2\pi a}{qZ_o} \left[-j\ \frac{P'(ka)}{P(ka)} - 2\ kgW\right], \tag{9}$$

where $W = j \sum_{s=1}^{\infty} \beta_s^{-1} = U + jV$, with

$$U = \sum_{s=1}^{S} (k^2a^2 - j_s^2)^{-1/2}, \ V = j \sum_{S+1}^{S_{max}} (j_s^2 - k^2a^2)^{-1/2}. \tag{10}$$

Here S is the number of the largest root of $J_o(x)$ below ka and S_{max} is a cutoff depending on g required to keep Eq. (10) from diverging.

The coupling impedance is defined as

$$Z(\omega) = -\frac{1}{q} \int_{-\infty}^{\infty} dz\ e^{jkz}\ E_z^I(z,r=0) = -\frac{2\pi}{q} F(k) \tag{11}$$

where the last form of Eq. (11) is obtained from Eq. (2). Using Eqs. (4) and (9) we then find, in the

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limit of small g,

$$Y(\omega) \equiv \frac{1}{Z(\omega)} = -\frac{q}{2gM} = \frac{\pi}{Z_o}\left[\frac{ja}{g}\frac{P'(ka)}{P(ka)} + 2ka\, W\right]. \quad (12)$$

If we assume $\Delta \equiv b-a << a$, we can approximate $P'(ka)/P(ka)$ to obtain our final result for a small pillbox:

$$Y(\omega) = \frac{\pi}{Z_o}\left[-\frac{ja}{kg\Delta} + 2ka(U + jV\right] \quad (13)$$

Figure 1 contains plots of R(ka), the real part, and X(ka), the imaginary part of $Z(\omega)$ obtained with our computer program[5] for $2g = .05a$, $\Delta = .1a$. These results agree with those of Henke[1] for the same parameters. The curves are exceedingly complex, showing among other features, the singular nature of the results whenever $ka = j_s$, that is, at the cutoff frequencies of the beam pipe. The results however appear much easier to interpret when we plot G(ka), the real part, and B(ka), the imaginary part of $Y(\omega)$, as in Fig. 2. In fact, the features of Eq. (13) appear to be very visible. Specifically, G(ka) reflects the steps which occur as ka passes each j_s, and B(ka) reflects the steps which occur just prior to ka passing j_s, as well as the smooth first term in Eq. (13).

One can draw further conclusions from Eq. (13). First, the terms containing the singular behavior for $ka = j_s$ are independent of g and Δ (except possibly for the logarithmic cutoff in S_{max}). Second, the smooth first term depends only on $g\Delta$ which is proportional to the volume of the pillbox. These features have been tested and generally confirmed by repeating the computation for small pillboxes of different size and shape.

If we consider smoothing the rapid fluctuations in the curves of Fig. 2, it is apparent that the result will correspond to a broad resonance. This can be done analytically from Eq. (13) by replacing the sum over s by an integral, the final result being

$$\frac{Z_o Y(\omega)}{\pi} = -\frac{ja}{kg\Delta} + 2kaW, \quad 2W \cong 1 + \frac{2j}{\pi}\,\ln\left(\frac{2\pi S_{max}}{ka}\right). \quad (14)$$

What Eq. (14) suggests is that we parametrize the impedance $Z(\omega)$ as

$$\bar{Z}(\omega) = \left[kaA + j\left(kaC - \left(\frac{D}{ka}\right)\right)\right]^{-1} \quad (15)$$

and obtain A,C,D from a least square fit of $|Z(\omega) - \bar{Z}(\omega)|^2$ to the numerical results. This has been done, with the result shown by the dashed curve in Fig. 1 for $2g = .05a$, $\Delta = .1a$, and in Fig. 3 for a semicircular obstacle of radius .1a. The dashed curve in Fig. 1 corresponds to $A = .00830$, $C = .022$, $D = 3.44$, and corresponds, according to Eq. (14), closely to the predicted values $A = 1/120$, $D = 10/3$. The predicted value of C depends on the cutoff S_{max} as well as the value of ka at the maximum.

The analysis for two pillbox obstacles proceeds in exactly the same way. The final result for $Z(\omega)$ is

$$\frac{\pi Z(\omega)}{Z_o} = \frac{y_1 + y_2 - 4kaT\cos kL}{y_1 y_2 - 4k^2a^2T^2}, \quad (16)$$

where L is the center to center separation of the obstacles, and

$$y_{1,2} = -\frac{ja}{kg_{1,2}\Delta_{1,2}} + 2kaW, \quad T = j\sum_{s=1}^{\infty}\frac{e^{-\beta_s L/a}}{\beta_s}. \quad (17)$$

We have here allowed the obstacles to have different dimensions.

For large L, the terms involving T and cos kL can be neglected and we obtain

$$Z(\omega) = \frac{Z_o}{\pi}\left[y_1^{-1} + y_2^{-1}\right], \quad (18)$$

corresponding to the sum of the impedances, as expected. For $L \to 0$ we have $T \cong W$, in which case

$$Y(\omega) = \frac{\pi}{Z_o}\left[\frac{-ja}{k(g_1\Delta_1 + g_2\Delta_2)} + 2kaW\right], \quad (19)$$

corresponding to a combined obstacle volume which is the sum of that for each one. The transition from Eq. (18) to Eq. (19), as governed by Eq. (16), contains the relevant information about the interference of the obstacles, as far as it affects the broad resonance.

In analogy with the "smoothing" of 2W in Eq. (16), we can also smooth T in Eq. (17) by converting the sum over s to an integral. In this case the sum becomes an integral representation of $H_o^{(2)}(z)$, leading directly to

$$2T \cong H_o^{(2)}(kL). \quad (20)$$

The remarkable feature of this result is that, for touching pillboxes, $L = 2g$, and, for $kg << 1$,

$$2kaT \to 2kaW \cong ka\, H_o^{(2)}(2kg) = ka\left[1 + \frac{2j}{\pi}\,\ln\frac{K}{kg}\right] \quad (21)$$

where $\ln K = .5772$ is Euler's constant, suggesting a cutoff of $S_{max} = 2Ka/2\pi g$ in Eqs. (10) and (14) for a pillbox of length 2g.

The parametrization of the interference is still quite complicated. The presence of the terms in T and cos kL in Eqs. (16) and (20) suggests that there are damped oscillations for large kL of the form $\exp(-2jkL)$ with relative amplitude proportional to $(8/\pi kL)^{1/2}$. Interference effects will therefore not be reduced to 10% until kL is of order 250. In those cases where the broad peak is located between ka = 5 and 10, interference is therefore expected to be important until L/a exceeds 25 to 50.

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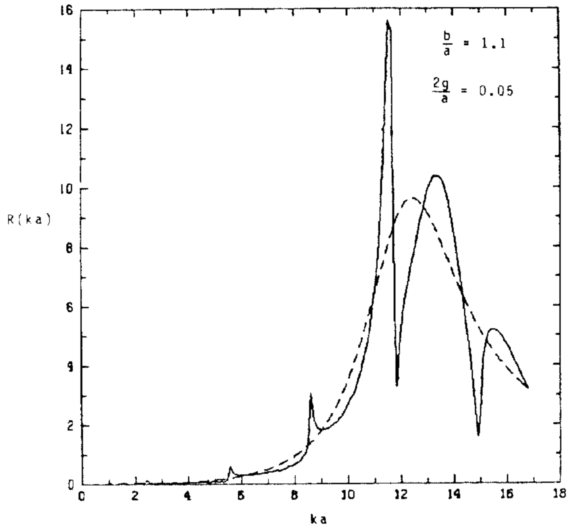

Fig. 1a. R(ka) vs ka for b = 1.1a, g = a/40.

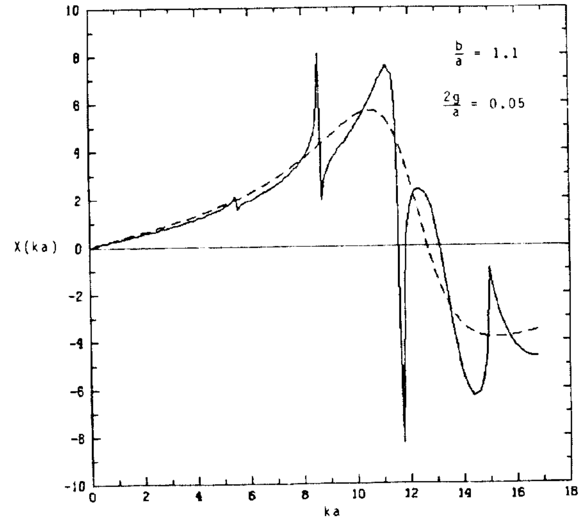

Fig. 1b. X(ka) vs ka for b = 1.1a, g = a/40.

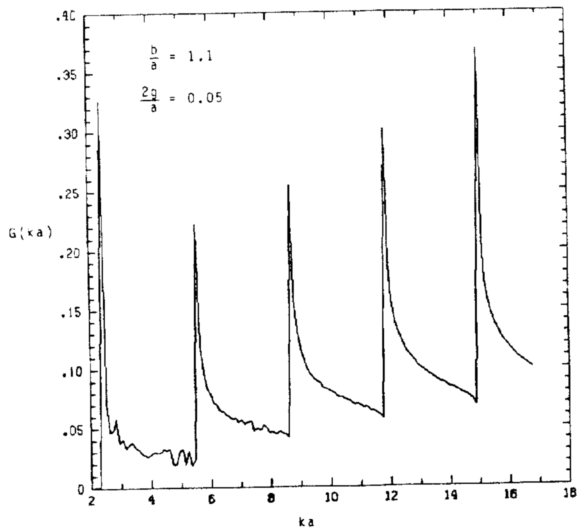

Fig. 2a. G(ka) vs ka for b = 1.1a, g = a/40.

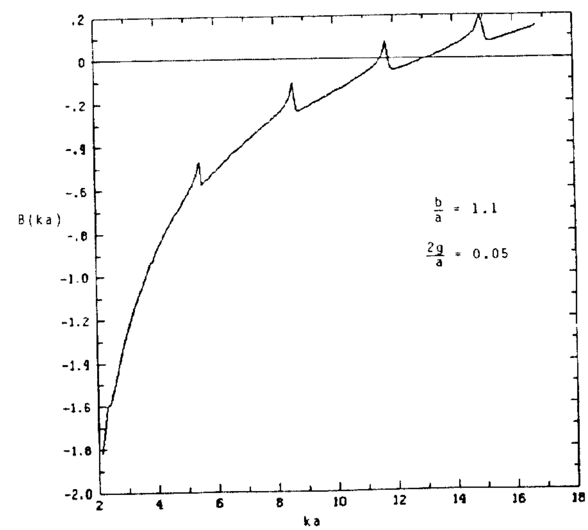

Fig. 2b. B(ka) vs ka for b = 1.1a, g = a/40.

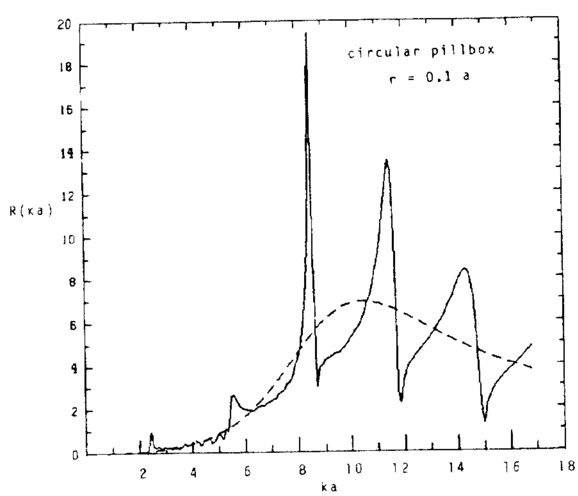

Fig. 3a. R(ka) vs ka for circular pillbox with radius 0.1a.

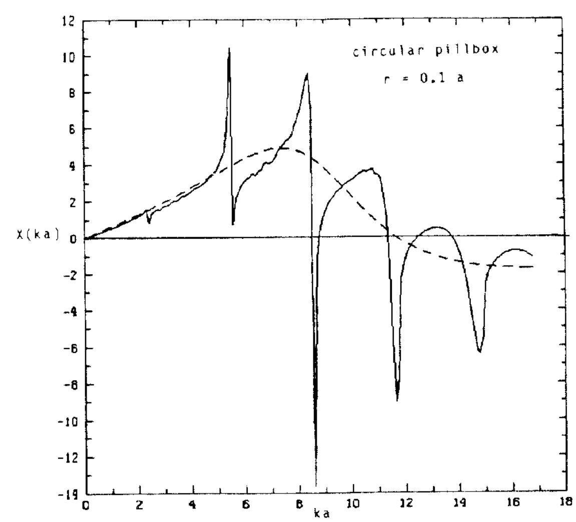

Fig. 3b. X(ka) vs ka for circular pillbox with radius 0.1a.