# LIE ALGEBRAIC METHODS FOR TREATING LATTICE PARAMETER ERRORS IN ACCELERATORS

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## Introduction

Lie algebraic methods as implemented in the code MARYLIE have proven successful for a variety of accelerator studies. Until now, these have not taken into account alignment and powering errors. Treatment of these errors will be briefly described here with more details given in reference 1.

We may consider a particle's motion to be completely described by 6 phase space variables $\zeta_i$ (i = 1, ..., 6): 3 coordinates and 3 canonical momenta. These variables will be a function of some independent variable z, and we indicate propagation of a particle from one value of z to a later one by application of the map $M$ to $\zeta$: $\bar{\zeta} = M\zeta$. We shall take as the coordinates X and Y, the two coordinates perpendicular to the beam's design trajectory, and T, the flight time. All these quantities, and their canonical momenta, are measured as a deviation from the design trajectory, and so are numerically small for particles close to it. We may therefore use a Lie algebraic perturbation series to represent the transfer map. The independent variable z we use is the longitudinal distance.

The maps are represented by the Poisson bracket Lie algebra. The Poisson bracket is defined by

$$[f,g] = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i}$$

where $x_i$, $p_i$ are the coordinates and canonical conjugate momenta of $\zeta$ respectively, and f and g are arbitrary functions on phase space. It has the property that $[\zeta_i,\zeta_j]$ is 0 if $\zeta_i$ and $\zeta_j$ are not canonical pairs, and 1 or -1 if they are. We may turn a function f into a Lie operator $:f:$, which is defined by its action on another function g, $:f:g = [f,g]$. These Lie operators also form a Lie algebra. More useful than the operators themselves are Lie transformations, the exponential of Lie operators;

$$e^{:f:} = I + :f: + \frac{1}{2} :f:^2 + \ldots$$

where I is the identity operator. It is possible to represent the map $M$ as a sequence of Lie transformations factored by order

$$M = e^{:f_1:}e^{:f_2:}e^{:f_3:}e^{:f_4:}\ldots, \qquad (1)$$

where $f_n$ is a homogeneous polynomial of order n in the coordinates and momenta. Note that the orders of the polynomials may be associated with the degree of the transformation. Because a Poisson bracket involves two derivatives we may conclude $[f_n,g_m] = h_{n+m-2}$. Therefore, for instance, $e^{:f_1:}\zeta$ is $\zeta$ plus a constant, $e^{:f_2:}\zeta$ is a linear transformation of $\zeta$, $e^{:f_3:}\zeta$ is $\zeta$ plus higher-order terms in the phase space variables. An example with $e^{:f_1:}$ is given below.

One may identify the first term in the sequence (1) with certain errors. If there are no errors, $f_1$ will be zero, but if there are alignment errors or mispowered dipoles, then $f_1$ will in general not be zero, and is proportional to the error. Alternatively, the first-order polynomial in the factorization is equivalent to the constant term in a Taylor expansion of $M$. If a quadrupole is misaligned, a particle entering on the design trajectory would feel a dipole field and would be bent off the design trajectory. Therefore, while the initial coordinates would be zero, the final coordinates would not, requiring that the Taylor series have a constant term, or the factored map have a first-order term.

While in general it is a lengthy calculation to obtain the factored map, it is possible to see how to write the map for a simple coordinate translation $\Delta X$ in the X (midplane) direction. Such a translation is given by $e^{-\Delta X:P_X:}$, i.e., $f_1 = -(\Delta X)P_X$ and $f_n = 0$, n > 1. If we calculate the effect of this transformation, we indeed verify that

$$\bar{X} = e^{-\Delta X:P_X:} X$$

$$= X - \Delta X [P_X,X] + \frac{1}{2}(\Delta X)^2 [P_X,[P_X,X]] \ldots = X + \Delta X,$$

i.e., the new X coordinate $\bar{X}$ differs from the original X by a constant $\Delta X$. Later sections give the outline of the calculation in the general case.

## Errors

The use of Lie algebraic techniques for a description and analysis of lattice parameter errors, i.e., small deviations in alignment and powering of elements, involves several major components. First, one must have an adequate description of the errors. For a mispowered element, one specifies, for instance, the field error $\Delta B/B$. For a misaligned element, on the other hand, the situation is more complicated, and is described below.

Once the description is available, one wants to generate the complete factored map of the form (1) from it. From all these maps, we may concatenate, or join, them into a single map of that form, representing a beamline or section. With the concatenated map, it is possible to track particles or find the fixed point and map nearby, giving lattice functions such as tune and chromaticity.

We address each of these topics separately in the following sections.

## Misalignment Description

In order to generate the map of a misaligned element, one first picks a particular point in the element, the fiducial point, and describes how it is positioned and oriented relative to how it should be positioned and oriented. This may be done by any parameterization of the Euclidean group. We have choosen three translations and three Euler angles. From these parameters describing the misalignment at the fiducial point, B, we need to determine the misalignment, C, again in terms of the Euclidean group, at the entry and exit faces. This may be done by simple geometric analysis and knowledge of the multi-

plication and inversion rules for the Euclidean group. If A is the Euclidean group element describing the change from the fiducial coordinates to the entry or exit face coordinates, we use the fact that this is the same in the ideally positioned magnet as the actual one to conclude that $C = ABA^{-1}$.

## Production of Maps

From the Euclidean group element describing the misalignment for the entry face and exit face, or from the Hamiltonian of a mispowered dipole, it is possible to produce a factored Lie algebraic map by a process analogous to that of producing an ideal element map from a Hamiltonian.[2] By assuming the error is small, we may conclude that the first-order term in the Hamiltonian is small; say that it is multiplied by a small quantity $\epsilon$. Specifically, we assume that the individual element errors contribute no more to the position and momentum errors of a particle than a particle may typically deviate in position and momentum from another in the bunch.

The process is iterative, with each step generating a new pseudo-Hamiltonian that has a first-order term one order higher in the small quantity $\epsilon$. Eventually, the iteration may be stopped because the remaining Hamiltonian may be neglected. The iteration is in powers of $\epsilon$; for $\epsilon = 0$, it has one step. Each step in the iteration produces a set of terms

$$N^{(i)} = e^{:g_2^{(i)}:} \, e^{:g_3^{(i)}:} \, e^{:g_4^{(i)}:} \, \cdots$$

on the right; when we are done we shall have a series of second-order-and-higher sets:

$$N = e^{:g_1:} \, N^{(n)} \, \cdots \, N^{(1)}$$

$$= e^{:g_1:} \, e^{:g_2^{(n)}:} \, e^{:g_3^{(n)}:} \, e^{:g_4^{(n)}:} \, \cdots$$

$$\times \, e^{:g_2^{(1)}:} \, e^{:g_3^{(1)}:} \, e^{:g_4^{(1)}:} \, \cdots$$

We may than concatenate the $N^{(i)}$ to form a single set

$$M = e^{:f_1:} \, e^{:f_2:} \, e^{:f_3:} \, e^{:f_4:} \, \cdots$$

with $f_1 = g_1$, using the techniques described below.

By studying the dynamics, it is possible to calculate what each of the $g_m^{(n)}$ are by integrating the pseudo-Hamiltonians $H^{(i)}$ and integrals of Poisson brackets of them. If it happens that the operator of the Hamiltonian, $:H:$, commutes with itself at different values of the independent variable, all the Poisson brackets go away and the calculation is straightforward. On the other hand, if it does not, the calculation can be quite involved.

For a mispowered dipole, one will then have the factored map. For a misaligned element, one need only calculate the factored map for the coordinate transformations at the entry and exit faces represented by C. They may be determined independent of specific knowledge of the element, save for the general statement of geometry given by the Euclidean group element A. These maps may be sandwiched around the ideal element map and then concatenated as described in the next section to produce a map for the misaligned element.

## Concatentation

Once these maps have been determined, it is desirable to concatenate them, that is, produce a single map corresponding, through a given order, to the combination of two or more maps

$$e^{:h_1:} \, e^{:h_2:} \, e^{:h_3:} \, \cdots$$

$$= e^{:f_1:} \, e^{:f_2:} \, e^{:f_3:} \, \cdots \, e^{:g_1:} \, e^{:g_2:} \, e^{:g_3:} \, \cdots \quad (2)$$

This includes feed-down effects due to the errors. Thus, for example to determine the map for a misaligned quadrupole, we would need to concatenate three maps, the coordinate transformation map at the leading face, the map for the ideal quadrupole, and the coordinate transformation map at the trailing face.

There are two tools available to derive the formula for the concatenation of maps. One is the Baker-Campbell-Hausdorff formula for combining exponentials: if

$$e^{:f:} e^{:g:} = e^{:h:}$$

then $:h: = :f: + :g: + \frac{1}{2} [f,g] + \cdots$, where each successive term is an operator of nested Poisson brackets of f and g. Applied to this calculation, we would join two exponentials into a single one, to the appropriate order, then separate them order by order in the right sequence, again using the Baker-Campbell-Hausdorff formula.

The other tool is the transformation rule

$$e^{:f(\zeta):} e^{:g(\zeta):} = e^{:g(e^{:f(\zeta):}\zeta):} \, e^{:f(\zeta):}$$

This is especially useful for second-order polynomials (linear maps) whose involvement in the Baker-Campbell-Hausdorff series would prevent its truncation by order.

Given no restrictions, the concatenation (2) would produce an infinite number of terms. In order to make this calculation tractable, one must assume the first-order polynomials are small in some sense. This will be the case for the Hamiltonian of an element with a small error as discussed above: a small factor $\epsilon$ multiplying the first-order term of the Hamiltonian will appear in the first-order term of the factored map. Then the presence of terms involving too many of these small quantities may be safely neglected. With this assumption, it is possible to derive concatenation rules to any desired order.

The general process for concatenation is as follows. First, we concentrate on moving the first order term to the left. That is, with the tools described above, rewrite the adjacent pair of transformations so that they are factored in the proper order,

$$e^{:f_n:} e^{:g_1:} = e^{:k_1:} \, e^{:k_2:} \, e^{:k_3:} \, \cdots \quad (3)$$

Initially, n will be the maximum order m of the polynomials, and because of this, we may write

$$e^{:f_m:} e^{:g_1:} = e^{:g_1:} \, e^{:f_m:}.$$

The next Lie transformation of order m-1 may not be as simple, however, and in general the polynomials $k_i$ are

not trivial. At each Lie transformation, the resultant $e^{:k_1:}$ is picked up for use with the next Lie transformation according to (3), while the $e^{:k_n:}$, $n >$ 2, are left behind for later. Eventually, all first-order terms will be towards the left, followed by the second and higher order terms. The first-order terms may be combined into a single exponential because any Poisson bracket is a constant. Next, the terms second order and higher may now be treated. First, the second-order terms are moved to the left by using the transformation rule. Then the higher-order terms are brought into proper sequence with the Baker-Campbell-Hausdorff formula as described above.

This process will work to any desired maximum order $m$. Currently, the calculation has been done through $m=6$, i.e., dodecupole order.

## Tracking

It is a straightforward matter to track particles with these factored maps; one first changes the phase space variables by adding constants corresponding to the transformation $e^{:f_1:}$. Then one applies the linear transformation $e^{:f_2:}$, specifically by taking a matrix dot product with the 6-vector of coordinates and momenta, because the linear transformation is actually stored as a matrix. From these, one may apply the higher-order parts of the map (polynomials of order three or greater) either by expanding to a certain number of terms the exponential to a series of Poisson brackets, or with a method that ensures the symplecticity of the result.

## Closed Orbit Determination

If one knows the closed orbit (or fixed point of the map) and the map around it, one can then apply a wide range of analysis tools that have been developed for Lie algebraic maps of machines without errors.[3] These include finding the tune, chromaticity and tune dependence with amplitude. The fixed point and adjacent map can be regarded as an alternate factorization of the map. Specifically, it is given by the polynomials $g_n$ where

$$M = e^{:f_1:} e^{:f_2:} e^{:f_3:} e^{:f_4:}$$

$$= e^{-:g_1:} e^{:g_2:} e^{:g_3:} e^{:g_4:} \ldots e^{:g_1:}$$

The transformation $e^{-:g_1:}$ maps 0 to the closed orbit, and the $g_n$, $n > 2$, give the map around that point.

These polynomials may be determined from the polynomials $f_n$ by one of two simple iterative schemes, one for the time independent case and one for the time dependent case.

Finally, these and related techniques are described in much more detail in reference 1. They have been coded into a working and tested code MARYLIE 3.1, which does such computations through order 4 in the polynomials (octupole order). A similar code, MARYLIE 5.1, accurate through order 6 (dodecupole order) has been coded and is being tested.

## References

1.  Healy, L.M., Ph.D. Dissertation, University of Maryland, August 1986, unpublished.

2.  Dragt, A.J. and E. Forest, J. Math. Phys. <u>24</u>, 2734 (1983).

3.  Forest, E., Ph.D. Dissertation, University of Maryland, August 1984, unpublished; E. Forest, Normal Form Method on Nonlinear Symplectic Maps, SSC-29, April 1985, unpublished report of the SSC Central Design Group, Berkeley, California.