PARTICLE TRACKING IN A SMALL ELECTRON STORAGE RING

K. Tsumaki
Energy Research Laboratory, Hitachi, Ltd.
1168 Moriyama-cho, Hitachi-shi, Ibaraki-ken, 316 Japan

Abstract

A particle tracking method for a ring system in which a sextupole magnetic field is distributed along the beam axsis has been developed. This method uses Jacobi's elliptic functions inside the bending magnet and the canonical integration method in the fringes. The calculation time for the new method is the same or faster than that of the canonical integration method, and it is ten times faster than the Runge-Kutta-Gill and thin lens approximation. A special characteristic of our method is that the calculation time is always constant, even if the magnet length in increased.

Introduction

Recently, synchrotron radiation has been considerd as a promising candidate for a soft X-ray source in micro lithography and several compact designs of electron storage rings have been proposed[1-3]. These employ bending magnets with a strong magnetic field and small curvature. Generally, such magnets have non-linear components in their magnetic field which cause reduction of the dynamic aperture for the circulating beam.[4-7]

There are several particle tracking methods for evaluation of the dynamic aperture. The thin lens approximation is the most popular. However, when the non-linear components is distributed along the beam axsis, such as in the bending magnet mentioned above, the problem must be treated as an assembly of many small sections which needs a long computation time. Ruth[8] has proposed a canonical integration method. This method is effective in shortening the computation time without deterioration in accuracy.

In actual magnets, a sextupole field is rather constant inside the magnets and change very much on the fringes. A particle equation of motion under a constant sextupole field can be solved analytically. Taking these conditions into acount, we propose a new method in which the analytical solution is applied inside the magnet and Ruth's method, on the fringes.

Tracking Method

Inside a bending magnet

This derivation is restricted to the horizontal betatron oscillation of a test particle under a sextupole field in a bending magnet and higher non-linear fields than the sextupole field, the radiation effect and electro magnetic interaction with the environment are neglected. Then, a particle equation of motion is given by the following equations,

$$\frac{d^2x}{ds^2} = -\frac{1}{\rho^2}x + s_e x^2$$
$$s_e = -\frac{s_{ex}}{B\rho} \qquad (1)$$

where ρ is a curvature of the particle trajectory in the magnet; B, strength of the magnetic field; and s_{ex}, the strength of the sextupole field. The solution of eq.(1) can be represented by Jacobi's elliptic functions sn and cn and is given by eq.(2).

$$x = a\,\mathrm{sn}^2(\omega s, k) + b\,\mathrm{cn}^2(\omega s, k) \qquad (2)$$

where a, b are betatron oscillation amplitudes; ω is angular frequency; and k, the modulus. Differentiating eq.(2) twice, the next equation is obtained.

$$\frac{d^2x}{ds^2} = \frac{2\omega^2(a^2-b^2+2abk^2+b^2k^2)}{a-b} - \frac{4\omega^2(a-b+ak^2+2bk^2)}{a-b}x + \frac{6\omega^2k^2}{a-b}x^2 \qquad (3)$$

The condition that eqs. (1) and (3) must be identical gives the following equations.

$$2\omega^2(a^2-k^2+2abk^2+b^2k^2)=0 \qquad (4)$$

$$\frac{4\omega^2\{(a-b)+(a+2b)k^2\}}{a-b} = \frac{1}{\rho^2} \qquad (5)$$

$$\frac{6\omega^2k^2}{a-b} = s_e \qquad (6)$$

One more equation to obtain four unknowns is given by the initial conditions for the position x_0 and momentum P_0 of the test particle at $s=s_0$. This gives eqs.(7) and (8).

$$x_0 = a\,\mathrm{sn}^2(\omega s_0, k) + b\,\mathrm{cn}^2(\omega s_0, k) \qquad (7)$$

$$p_0 = 2\omega(a-b)\,\mathrm{sn}(\omega s_0, k)\,\mathrm{cn}(\omega s_0, k)\,\mathrm{dn}(\omega s_0, k) \qquad (8)$$

From eqs.(7) and (8), eq.(9) is obtained.

$$\omega^2 = \frac{(a-b)p_0^2}{4(x_0-a)(x_0-b)\{k^2(x_0-b)-(a-b)\}} \qquad (9)$$

From eqs.(4), (5), (6) and (9), the following equations are deduced.

$$2s_e\rho^2(a^2+b^2+ab)-3(a+b)=0 \qquad (10)$$

$$2s_e(x_0^2a^2-a^2b^2+x_0^2ab+x_0^2b^2)+(3p_0^2-2s_ex_0^3)(a+b)=0 \qquad (11)$$

$$k^2 = -\frac{(a^2-b^2)}{(2a+b)b} \qquad (12)$$

$$\omega^2 = \frac{(a-b)s_e}{6k^2} \qquad (13)$$

Solving eqs.(10) to (13), a, b, ω and k are obtained. The position and momentum of a particle at $s=s+s_0$ are given by eqs.(14) and (15).

$$x = a\,\mathrm{sn}^2(\omega(s+s_0), k) + b\,\mathrm{cn}^2(\omega(s+s_0), k) \qquad (14)$$

$$p = 2\omega(a-b)\,\mathrm{sn}(\omega(s+s_0), k)\,\mathrm{cn}(\omega(s+s_0), k)\,\mathrm{dn}(\omega(s+s_0), k) \qquad (15)$$

Equations (14) and (15) are the basic equations for particle tracking. It is not practical from the viewpoint of computatational time to get s_0 from the solution of eq.(7). Instead, eqs.(16) and (17) are used; they are deduced from eqs.(14) and (15).

CH2387-9/87/0000-0467 $1.00 © IEEE
PAC 1987

$$x=\left\{a\left(\mathrm{sn}^2\omega s\cdot \mathrm{cn}^2\omega s_0\cdot \mathrm{dn}^2\omega s_0+\mathrm{cn}^2\omega s\cdot \mathrm{dn}^2\omega s\cdot \mathrm{sn}^2\omega s_0\right)+b\left(\mathrm{cn}^2\omega s\cdot \mathrm{cn}^2\omega s_0+\mathrm{sn}^2\omega s\cdot \mathrm{dn}^2\omega s\cdot \mathrm{sn}^2\omega s_0\cdot \mathrm{dn}^2\omega s_0\right)+\frac{\mathrm{sn}\omega s\cdot \mathrm{cn}\omega s\cdot \mathrm{dn}\omega s\cdot p_0}{\omega}\right\}\frac{1}{(1-k^2\mathrm{sn}^2\omega s\cdot \mathrm{sn}^2\omega s_0)^2} \tag{16}$$

$$\begin{aligned}p=\Big\{&2\omega(a-b)\left(\mathrm{sn}\omega s\cdot \mathrm{cn}\omega s\cdot \mathrm{dn}\omega s\cdot \mathrm{cn}^2\omega s_0\cdot \mathrm{dn}^2\omega s_0\right.\\&-\mathrm{sn}\omega s\cdot \mathrm{cn}\omega s\cdot \mathrm{dn}^3\omega s\cdot \mathrm{sn}^2\omega s_0\cdot \mathrm{dn}^2\omega s_0\\&\left.-k^2\,\mathrm{sn}\omega s\cdot \mathrm{cn}^3\omega s\cdot \mathrm{dn}\omega s\cdot \mathrm{sn}^2\omega s_0\cdot \mathrm{cn}^2\omega s_0\right)\\&+p_0\left(k^2\mathrm{sn}^2\omega s\cdot \mathrm{cn}^2\omega s\cdot \mathrm{dn}^2\omega s\cdot \mathrm{sn}^2\omega s_0\right.\\&-k^2\,\mathrm{sn}^2\omega s\cdot \mathrm{cn}^2\omega s\cdot \mathrm{cn}^2\omega s_0\\&\left.+\mathrm{cn}^2\omega s\cdot \mathrm{dn}^2\omega s-\mathrm{sn}^2\omega s\cdot \mathrm{dn}^2\omega s\cdot \mathrm{dn}^2\omega s_0\right)\\&+\frac{p_0^2\,k^2\,\mathrm{sn}^3\omega s\cdot \mathrm{cn}\omega s\cdot \mathrm{dn}\omega s}{2\omega(a-b)}\Big\}\frac{1}{(1-k^2\mathrm{sn}^2\omega s\cdot \mathrm{sn}^2\omega s_0)^3}\end{aligned} \tag{17}$$

where modules k is omitted for simplicity. $\mathrm{Cn}(ws_0,k)$ and $\mathrm{sn}(ws_0,k)$ can be obtained from eq.(7).

$$\mathrm{sn}^2(\omega s_0,k)=\frac{x_0-b}{a-b} \tag{18}$$

$$\mathrm{cn}^2(\omega s_0,k)=-\frac{x_0-a}{a-b} \tag{19}$$

Fringe

The third order canonical integration method[4] which is obtained by a Tayler expansion of a Hamiltonian is applied in the fringe. The third order map is given by the following scheme:[4]

$$H=g(p)+v(x,s)\quad,\quad f=-\frac{\partial v}{\partial x} \tag{20}$$

$$\mathrm{Map}:(x,p)=M_3(h)(x_0,p_0)$$

given by three step process

$$\begin{aligned}p_1&=p_0+c_1hf(x_0,s_0)\ , & x_1&=x_0+d_1h\frac{dg(p_1)}{dp}\\ p_2&=p_1+c_2hf(x_1,s_0+d_1h)\ , & x_2&=x_1+d_2h\frac{dg(p_2)}{dp}\\ p&=p_2+c_3hf(x_2,t_0+(d_1+d_2)h)\ , & x&=x_2+d_3h\frac{dg(p)}{dp}\end{aligned} \tag{21}$$

where c_i, $d_i(i=1,2,3)$ must satisfy the next conditions.

$$\begin{aligned}&c_1+c_2+c_3=1\ ,\quad d_1+d_2+d_3=1\ ,\quad c_2d_1+c_3(d_1+d_2)=\frac{1}{2}\\&c_2d_1^2+c_3(d_1+d_2)^2=\frac{1}{3}\ ,\quad d_3+d_2(c_1+c_2)^2+d_1c_1^2=\frac{1}{3}\end{aligned} \tag{22}$$

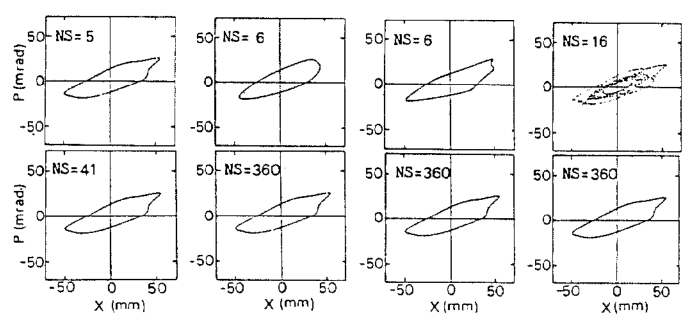
Results and Discussion

As an example, we applied our method to a racetrack storage ring[1] with 1.4m-long bending magnets. Canonical integration method, thin lens approximation and the Runge-Kutta-Gill method were used for comparison with the new method. In the calculation, the bending magnet was assumed to be composed of small sections with length h. (Hereafter, each section is called a step.) In the canonical integration method, eq.(21) was used. In the thin lens approximation, the following three-step procedure was used.

$$\begin{aligned}&\begin{pmatrix}x_1\\p_1\end{pmatrix}=\begin{pmatrix}\cos\frac{h}{2\rho} & \rho\sin\frac{h}{2\rho}\\ -\frac{1}{\rho}\sin\frac{h}{2\rho} & \cos\frac{h}{2\rho}\end{pmatrix}\begin{pmatrix}x_0\\p_0\end{pmatrix}\\&x_2=x_1\ ,\quad p_2=p_1+\frac{h\,s_{ex}}{B\rho}x_1^2\\&\begin{pmatrix}x\\p\end{pmatrix}=\begin{pmatrix}\cos\frac{h}{2\rho} & \rho\sin\frac{h}{2\rho}\\ -\frac{1}{\rho}\sin\frac{h}{2\rho} & \cos\frac{h}{2\rho}\end{pmatrix}\begin{pmatrix}x_2\\p_2\end{pmatrix}\end{aligned} \tag{23}$$

In the Runge-Kutta-Gill method, the usual fourth order method was employed.

Fig.1 shows the mapping results on a phase space. The map shape by the new method does not depend on the number of steps. The plotted tracking resuls from the Runge-Kutta-Gill method do not show a closed curve on the phase space, because the symplectic condition is not satisfied in this method. Then, further comments must be made on the canonical integration method and the thin lens approximation. As both methods satisfy the symplectic condition, Liouville's theorem holds and particles continue to circulate without damping of the betatron oscillation, whatever the step size is. However, in the case of a large step size, the map shape and the area enclosed by tracks on the phase space differ from those of the small step size. For example, in the canonical integration method, the maximum value of the position x_{max} is 54mm and the minimum value x_{mini} is -50mm when the number of steps is large. Contrary to this, x_{max} is 44mm and x_{mini} is -45mm when the number of steps is small. Namely, the aperture is reduced about 20% when the step size is large. This shows that the correct dynamic aperture cannot be obtained if the step size is large, even if the symplectic condition is fulfilled.



(a) This work (b) Canonical integration method (c) Thin lens approximation (d) Runge-Kutta-Gill method

Fig.1 Mapping results on a phase space. (NS : number of steps)

Next, we investigated the relationship between the number of steps and accuracy of the final position for a particle. Fig.2 results the number of steps in a bending magnet and the position of a particle after circulating ten thousand turn in a storage ring. Our method gives a constant value which is independent of the number of steps. Final positions by other tracking methods depend on the number of steps used in the calculation. The final position by the canonical integration method oscillates when the number of steps is less than 50. Results of the thin lens approxiamtion and the

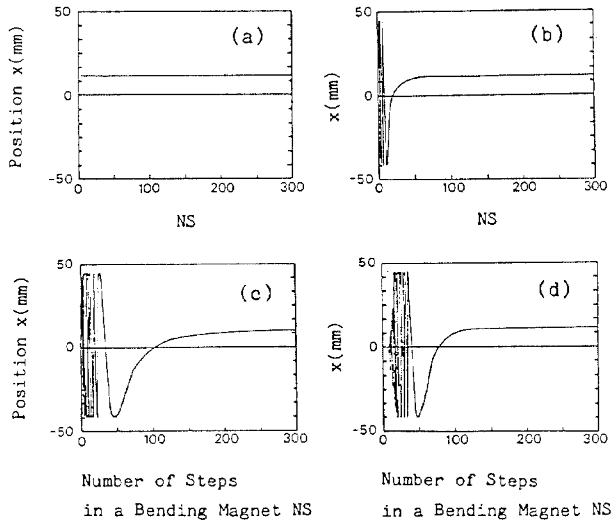


Fig.2 Relation between the number of steps and the position after ten thousand turns ((a):This work; (b):Canonical integration method; (c):Thin lens approximation; (d):Runge-Kutta-Gill method)

Runge-Kutta-Gill method oscillate if the number of steps is less than 150 and 100, respectively. This is then the reason as to why these other methods do not give the correct mapping results on the phase space if the number of steps is small, as shown in Fig.1.

Fig.2 shows that a large number of steps must be taken in the bending magnet to obtain correct results. The calculation time increases as the number of steps increases. Therefore, we studied the relation between the accuracy δx, defined by $\delta x = |(x-x_{\infty})/x_{\infty}|$ and the calculation time. The results are shown in Fig.3, where x_{∞} is the value which converges within 0.1% by decreasing the step size in each method. The computer used for the calculation was the HITAC M200H. The accuracy of our method is always within 1%. Relative calculation time to obtain 1% accuracy is shown in Table 1 for two cases. For the case of 1.4m-long magnet, our method has almost the same calculation

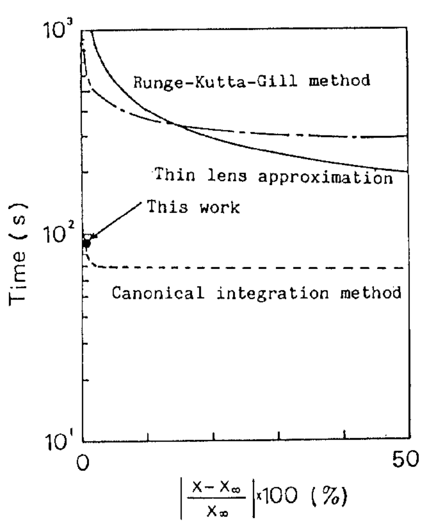


Fig.3 Relation between the accuracy $\delta x = |(x-x_{\infty})/x_{\infty}|$ and the calculation time.

Table 1 Relative calculation time to obtain 1% accuracy (l_B :bending magnet length)

	l_B =1.4m	l_B =14m
This work	1	1
Canonical Integration Method	1	3
Thin Lens Approximation	14	—
Runge-Kutta-Gill	9	—

time as the canonical integration method and is about ten times faster than the Runge-Kutta-Gill method and the thin lens approximation. For the case of 14m-long magnet, calculation time of our method is the same as for the case of 1.4m-long magnet and three times faster than the canonical integration method.

A special characteristic of our method is that the calculation time is always constant, without regard to the magnet length, though the calculation time of other methods increases with the magnet length.

Conclusions

A tracking method has been developed under a strong sextupole field in a bending magnet. This method uses Jacobi's elliptic functions inside the bending magnet and the canonical integration method in the fringes. The method was applied to a small electron storage ring and accuracy within 1% was obtained after ten thousand turn. Comparison with the canonical integration method, the thin lens approximation and the Runge-Kutta-Gill method showed that the calculation time by our method was the same or faster than that of the canonical integration method and ten times faster than the Runge-Kutta-Gill and thin lens approximation. A special characteristic of our method is that the calculation time is always constant, even if the magnet length is increased.

Acknowledgements

The auther is greatly indebted to Prof. K.Kamiya of KEK for his valuable suggestions and discussions. Thanks are also due to Prof. M.Kihara of KEK for many useful suggestions. Finally, appreciation is due to Drs. M.Nishi and N.Ozaki for their continuous encouragement.

References

(1) A.Heuberger, "X-RAY LITHOGRAPHY", Proc. of Microcircuit Engineering, September 23-25, 1986
(2) F.C.Younger, A.Jackson,K.D.Jenkins,P.F.Meads,L.S.B.Ng, "Electron Storage Ring for x-ray Lithography", Proceedings of the 1987 Particle Accelerator Conference, Washington, D.C. .
(3) G.A.Decker, B.C.Craft, "Optimization of Compact Synchrotron Optics for X-Ray Lithography", Proceedings of the 1987 Particle Accelerator Conference, Washington,D.C. .
(4) R.D.Ruth, "A CANONICAL INTEGRATION TECHNIQUE", IEEE Trans. Nucl.Science,Vol.NS-30, No4, August, 1983, pp2669-2671.
(5) H.Wiedemann, PEP Technical Memo 230, 1981
(6) K.L.Brown, D.C.Carey, F.C.Iselin and F.Rothacker, "TRANSPORT, A Computer Program for Designing Charged Particle Beam Transport Systems", CERN 80-4.
(7) R.Servranckx,"Depart:A Differential Equation Particle Tracking Program", Particle Accelerators, vol.6, pp83-93, 1975.