THERMAL LIMITS ON WAVE AMPLITUDE IN THE PLASMA WAKE-FIELD ACCELERATOR

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Abstract

We discuss the nonlinear regime of the Plasma Wakefield Accelerator and its potential advantages over linear operation, and describe the general behavior of the plasma electron phase space in large amplitude plasma waves. We develop a Hamiltonian formalism for calculating the trapping of thermal electrons from the background plasma, and derive approximate limits on plasma wave amplitude due to loading by trapped electrons and other thermal effects.

Introduction

The one-dimensional nonlinear theory of the Plasma Wakefield Accelerator (PWFA), developed recently by Rosenzweig,[1,2] Ruth *et al.*[3] and Amatuni *et al.*,[4] predicts certain advantages the extremely nonlinear regime. It has been shown that the transformer ratio, the ratio of the maximum decelerating field inside the driving electron bunch to the maximum accelerating field found in the wake of the driver, is enhanced by driving the plasma waves with an electron bunch of density one-half that of the plasma electrons. This method of obtaining large transformer ratios is asserted to be more straightforward than alternatives in the linear regime, and also to lessen the multiple scattering that the accelerating particles undergo in a PWFA, as the plasma used need not be as dense.

The high transformer ratios obtained in this nonlinear scheme depend on driving the plasma electron density waves to a extremely large amplitudes. In contrast to the linear waves, the maximum energy the electrons reach during the oscillation is many times their rest mass energy. The electron oscillation period is an increasing function of this maximum energy. As the amplitude of the wave grows, the wave steepens dramatically, and the positive excursion in density becomes very large in amplitude and narrow in time. The combination of these two effects suggest that nonlinear plasma waves will be very sensitive to any thermal effects, as the plasma electron phase space dynamics can be strongly affected by the velocity distribution prior to excitation. The relativistic velocities attained by the plasma electrons are close to the phase velocity of the wave. Therefore, we must consider the possibility of trapping of the thermal electrons in the wave potential well. Wave damping due to the effects arising from the initial thermal distribution also requires investigation.

The purpose of this paper is to derive, using the nonlinear fluid theory formalism of Refs. 1-3 as well previous work on free nonlinear oscillations by Noble,[5] the conditions for trapping of thermal plasma electrons in the potential well of the large amplitude plasma wave excited by a driving beam. We then obtain an estimate on the saturation amplitude of the plasma wave due to trapping by considering the energy balance between the wave and the trapped electrons. Also starting with the fluid approximation, we consider an alternative estimate on the maximum wave amplitude due to the effects of the non-zero electron temperature on the wave motion itself.

Nonlinear plasma oscillations

We first recapitulate the major results of the one-dimensional nonlinear fluid theory. We assume that the plasma ions form an immobile neutralizing background of density n_0 and define the plasma electron density n, velocity $v = \beta c$, and the plasma frequency $\omega_p = (\frac{4\pi e^2 n_0}{m_e})^{1/2}$. By writing the fully relativistic fluid equations for the plasma electrons in the presence of an ultra-relativistic beam of density n_b and velocity $v_b = \beta_b c$ and assuming the wave motion is a function only of the variable $\tau = \omega_p(t - z/v_b)$ we obtain the result of Akhiezer and Polovin[6] :

$$n = \frac{n_0 \beta_b}{\beta_b - \beta}, \tag{1}$$

$$\frac{d^2}{d\tau^2}\left[\frac{1-\beta_b\beta}{(1-\beta^2)^{1/2}}\right] = \beta_b^2\left[\frac{\beta}{\beta_b - \beta} + \frac{n_b}{n_0}\right]. \tag{2}$$

In the case we consider $\beta_b \to 1$ and the equation simplifies, with the substitution of a new independent variable $x = (\frac{1-\beta}{1+\beta})^{1/2}$, to

$$x''(\tau) = \frac{1}{2}\left[\frac{1}{x^2} - 1 + \frac{2n_b}{n_0}\right], \tag{3}$$

where the prime indicates differentiation with respect to τ. When n_b is a constant, zero or nonzero, the first integral of Eq. (3) is obtained trivially. Since we are interested primarily in the dynamics of the free oscillations behind the driving beam, we concentrate on the integral of the homogeneous equation, which is of the form

$$\left(x'(\tau)\right)^2 = 2\gamma_m - (x + \frac{1}{x}), \tag{4}$$

where γ_m is the maximum Lorentz factor the fluid electrons attain due to the wave motion. The electron Lorentz factor is thus given as a function of x by $\gamma = (x + x^{-1})/2$.

The solution of Eq. (4) has been found in terms of elliptical integrals,[5] so we may in principle calculate x exactly as a function of τ. The solutions are discussed in great detail in Refs. 2and 5, and the reader is urged to consult these sources for a full description. It is not necessary to repeat these previous results for our purposes, as the electric field can be deduced directly from the fluid equations,[2] in terms of Eq. (4)

$$E(x) = -(m_e c\omega_p/e)x' = \pm(m_e c\omega_p/e)\ \left[2\gamma_m - (x + \frac{1}{x})\right]^{1/2}. \tag{5}$$

Thus we may identify x as being proportional to the electrostatic potential and form the potential energy function

$$V = -e\phi(x) = mc^2(1 - x). \tag{6}$$

The maximum of the potential energy occurs at the minimum in x.

Trapping of thermal electrons

In order to study the trapping of the thermal background electrons in this potential well, we employ the Hamiltonian formalism used by Ruth and Chao[7] in their discussion of the Plasma Beatwave Accelerator (PBWA). The Hamiltonian may be written formally as

$$H = -e\phi(z - v_b t) + c\left[m^2c^2 + p^2\right]^{1/2}. \tag{7}$$

We convert the Hamiltonian to a constant of the motion using the generating function

$$F_2(z, p^*, t) = (z - v_b t)p^* \tag{8}$$

to give a new coordinate, momentum and Hamiltonian:

$$z^* = z - v_b t = -\frac{v_b}{\omega_p}\tau, \qquad p^* = p \tag{9}$$

and

$$H^* = H - v_b p^* = mc^2\big(1-x(z^*)\big) + c\big[m^2c^2+p^2\big]^{1/2} - v_b p^*. \tag{10}$$

The Hamiltonian is now a useful invariant.

In order to see if an electron may be picked out of the tail of the thermal distribution of the background plasma and trapped in the potential well (*bucket* in accelerator physics parlance) of the wave, we evaluate H^* before the arrival of the driving beam. An electron of initial forward velocity $v = \beta_0 c$ has an invariant

$$H_0^* = mc^2\gamma_0(1-\beta_b\beta_0), \tag{11}$$

where $\gamma_0 = (1-\beta_0^2)^{-1/2}$. The value of H^* at the separatrix between trapped and untrapped electrons is found by evaluating the Hamiltonian at the unstable fixed point at the back of the bucket, where $E(x) = 0$, $x = x_{min}$ and $\gamma = \gamma_b$. Substitution of these values yields

$$H_{sep}^* = mc^2\left[1 - x_{min} + \gamma_b^{-1}\right] \tag{12}$$

Assuming that the plasma is cold ($\beta_0 \ll 1$), the oscillation amplitude is large ($x_{min} \ll 1$) and the driving beam is ultra-relativistic ($\gamma_b \gg (x_{min})^{-1}$), the condition for trapping, $H_0^* \leq H_{sep}^*$, can be written,

$$\beta_0 \geq x_{min} \simeq (2\gamma_m)^{-1}. \tag{13}$$

Wave loading due to background trapping

We assume that the background plasma electrons initially have a Gaussian velocity distribution characterized by a rms thermal velocity $v_{th} = \beta_{th}c = (kT_e/m)^{1/2}$. The fraction of the plasma electrons trapped ϵ_{tr} is, assuming the minimum trapping velocity $\beta_{tr} \gg \beta_{th}$,

$$\epsilon_{tr} = erfc\Big(\frac{\beta_{tr}}{\sqrt{2}\beta_{th}}\Big) \simeq \Big(\sqrt{\frac{8}{\pi}}\beta_{th}\gamma_m\Big)e^{-2(2/\beta_{th}\gamma_m)^2}. \tag{14}$$

To estimate the the saturation amplitude of the wave we examine the rate of energy input to the wave by the driving beam versus the rate of energy extraction by the trapped electrons. The rate of energy per unit area u per unit length input to the wave can be shown to be[8]

$$\frac{du}{dz}\,|_{in} = (\gamma_m - 1)n_0mc^2. \tag{15}$$

The question remains as to how much energy the trapped electrons take out of the wave. If they all stay inside the bucket then the wave is destroyed in a short distance. This cannot happen, of course, because of the self-consistent bucket loading that has not been accounted for yet. This is a difficult calculation, but it is qualitatively clear that the trapped electron density builds up at the back of the bucket, as the rate of longitudinal phase advance $\nu = |\frac{dz^*}{dt}|$ is very small as the unstable fixed point is approached. The trapped electrons provide a self-decelerating wake, and the vast majority of them are detrapped after gaining an energy less than or equal to the beam energy. Some of the initially more energetic trapped electrons will undoubtedly remain inside the bucket, however, and accelerated to very high energy.

Thus the energy extraction rate due to trapping is

$$\frac{du}{dz}\,|_{out} = \epsilon_{tr}n_0\gamma_b mc^2 \tag{16}$$

and the net energy input to the wave per unit area per unit length is

$$W = \frac{du}{dz}\,|_{in} - \frac{du}{dz}\,|_{out} = (\gamma_m - \gamma_b\epsilon_{tr} - 1)n_0mc^2. \tag{17}$$

We obtain the saturation amplitude of the wave due to trapping by taking $\partial W/\partial\gamma_m = 0$. The resulting transcendental equation can be solved for the amplitude parameter γ_m approximately in the limit that $\gamma_b \gg \alpha_{th}$;

$$\gamma_m \simeq F(\gamma_b/T_e)\left[\frac{mc^2}{kT_e}\right]^{1/2} \tag{18}$$

where $F(\gamma_b/T_e)$ is a very slowly varying function that is the neighborhood of 0.25. For a PWFA operating in the nonlinear regime the maximum normalized (to the linear "wave-breaking limit") accelerating field is then

$$\frac{eE_m}{mc\omega_p} \simeq (2\gamma_m)^{1/2} \simeq \left[\frac{mc^2}{2kT_e}\right]^{1/4}. \tag{19}$$

As a numerical example we take $kT_e = 10\ eV$ and calculate a maximum electric field of approximately ten times wave-breaking.

Thermal limits on wave amplitude

The previous analysis is based on the assumption that the equations we have written in the fluid approximation describe the salient features of the wave motion and, in doing so, have neglected the possible collective effects of the thermal electron distribution. Here we develop a test for the validity of our assumption, and discuss the physical mechanisms of wave saturation by thermal damping.

One might anticipate from the previous analysis that very small differences in the initial velocity of the plasma electrons yield large differences in their energy at the back of the bucket. This is in fact the case, as can be shown easily from Eq. (10);

$$\Delta\gamma\,|_{x_{min}} = \frac{\partial\gamma}{\partial x}\Delta x \simeq \frac{\beta_{th}}{x_{min}^2} \simeq 4\beta_{th}\gamma_m^2. \tag{20}$$

This energy spread indicates an uncertainty in the rate of phase advance ν of the plasma electrons which is very detrimental in a wave with such steepened density profiles. Again using Eq. (10) we have

$$\nu = \frac{\partial H^*}{\partial p^*} = \frac{-2x^2}{x^2+1}. \tag{21}$$

Using Eq. (21) we find the spread in ν

$$\frac{\Delta\nu}{\nu} = \frac{4\beta_{th}}{x(x^2+1)}. \tag{22}$$

The largest uncertainty in phase advance occurs in the neighborhood of the minimumin x, i.e. the back of the bucket. The total spread in phase (time) accumulated over the oscillation period T must be narrower than the width a of the steep excursion in density near positive turning point (at the back of the bucket), or the accelerating field will be damped. The fractional half width a/T of this spike can be estimated easily by noting the approximate functional form of the steepened waves (cf. Ref. 2),

$$\frac{a}{T} \simeq \frac{1}{4\gamma_m^2}. \qquad (23)$$

The fractional spread in τ at the maximum in the accelerating field is approximately

$$\frac{\Delta\tau}{\tau} \simeq \frac{\Delta x}{x'\omega_p T} \simeq \sqrt{\frac{2}{\gamma_m}}\left[\frac{\beta_{th}}{\omega_p T}\right]. \qquad (24)$$

Using Noble's[9] result for the period in the nonlinear limit,

$$\omega_p T \simeq 4\sqrt{2\gamma_m}, \qquad (25)$$

we find

$$\frac{\Delta\tau}{\tau} \simeq \frac{\beta_{th}}{4\gamma_m}. \qquad (26)$$

Comparing Eqns. (23) and (26)we obtain the maximum amplitude

$$\gamma_m \simeq \left[\frac{mc^2}{kT_e}\right]^{1/2}, \qquad (27)$$

in close agreement with Eq. (18).

Thus the inherent momentum spread detunes the motion of the plasma electrons and is a cause of phase spread in the wave. The initially forward going plasma electrons are advanced in phase and can damp the accelerating field, reducing its maximum amplitude by rounding the top of the sawtooth shaped electric field profile. The saturation mechanism discussed here is the incoherence of the electron oscillations, and is therefore a form of Landau damping.

Discussion

The two estimates presented here are derived from appeals to distinct physical effects, trapping and thermal damping. The fortuitous agreement of the alternative methods for estimating of the maximum plasma wave amplitude can be interpreted as a check on the internal consistency of our approach. Obviously, something goes awry with the waves as we have modeled them as the amplitude nears $\gamma_m \simeq 1/\beta_{th}$, and it is our view that this is the correct functional dependence of the maximum amplitude on the temperature. The thermal damping may be the more serious constraint, as the incoherency can build up over more than one oscillation behind the bunch. We should view these calculations as upper estimates on the actual physical wave amplitude. Our analysis is essentially a perturbation treatment which assumes the one-dimensional fluid solutions are valid. There may be some subtlety that is overlooked by this method. A more self-consistent analysis aided by some computer simulation work would help answer these questions. Also, we have entirely ignored the three-dimensional effects in our discussion, albeit because the nonlinear formalism has not as yet been developed to allow transverse variations is the plasma waves. These effects certainly will have an impact on the maximum wave amplitude in a real PWFA.

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