SCALING LAWS FOR RFQ DESIGN PROCEDURES*

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Summary

Scaling laws are relations between accelerator parameters (electric field, rf wavelength etc.) and beam parameters (current, energy, emittance, etc.) that define surfaces of constant accelerator performance in parameter space. These scaling laws can act as guides for designing radio-frequency quadrupoles (RFQs). We derive several scaling relations to show the various tradeoffs involved in choosing RFQ designs and to provide curves to help choose starting points in parameter space for optimizing an RFQ for a particular requirement. We show that there is a unique scaling curve, at a synchronous particle phase of -90°, that relates the beam current, emittance, particle mass, and space-charge tune depression with the RFQ frequency and maximum vane-tip electric field, provided that we assume equipartitioning and equal longitudinal and transverse tune depressions. This scaling curve indicates the maximum performance limit one can expect at any point in any given RFQ. We show several examples for designing RFQs using this procedure.

Introduction

We define a procedure for obtaining initial RFQ designs.[1] Scaling laws, derived below, are used to obtain an initial estimate of the RFQ parameter regime to satisfy the beam-dynamics requirements. RFQ optimization can then be done using program RFQDES,[2] which is a general-purpose RFQ design program that allows maximum flexibility in choosing RFQ design algorithms.

We start by writing the linear space-charge force parameters for a uniform charge-density ellipsoid, then list the RFQ Mathieu equation parameters (the Mathieu equation approximately describes particle motion in the RFQ), and finally combine the RFQ and space-charge Mathieu terms to form scaling laws. Scaling-law plots, made to facilitate RFQ designs, are given, and we show several examples of RFQ designs using these laws. We list below some of the parameters.

A_T = transverse normalized RFQ acceptance/λ
a = RFQ minimum vane radius
β_s = synchronous velocity/c
$\Delta_{T(L)}$ = transverse (longitudinal) space-charge-force parameter
E_0 = maximum electric field on the RFQ vane tip at quadrupole symmetry = $V\sqrt{\chi}/a$
$\epsilon_{T(L)}$ = total transverse (longitudinal) normalized emittance/λ
g = $\sqrt{\epsilon_T\sigma_T/\epsilon_L\sigma_L}$ [Eq. (14)]
h = $A_T/\epsilon_T = a^2/R_M^2$
I = beam current in amperes
I_0 = modified Bessel function
$\sigma_{To(Lo)}$ = transverse (longitudinal) phase advance per period (zero current)
$\sigma_{T(L)}$ = $\sigma_{To(Lo)}\sqrt{1-\mu_{T(L)}} = \sqrt{\sigma^2_{To(Lo)} - \Delta_{T(L)}}$ space-charge depressed, transverse (longitudinal), phase advance per period
λ = free-space rf wavelength of the RFQ
M_0 = beam particle's rest-mass energy in electron volts
m = RFQ modulation parameter = (maximum/minimum) vane radius
$\mu_{T(L)}$ = transverse (longitudinal) space-charge tune depression parameter
φ_s = synchronous phase (normally negative)
Ψ = flutter factor = (maximum/minimum) beam radius
R_M, R'_M = maximum beam radius and dR/ds (s is the distance along the trajectory/$\beta\lambda$)
V = RFQ vane voltage
Z_M, Z'_M = maximum beam length/2 and dZ/ds (s is the distance along the trajectory/$\beta\lambda$)
Z_0 = impedance of free space = 376.73 Ω

Below, all lengths are divided by the rf wavelength λ and all electric potentials and fields and the current are divided by the particle's rest-mass energy in electron volts. Consequently, all parameters that appear in the following equations are unitless and the equations do not explicitly contain λ and Mo. We will return to SI units after we derive the scaling laws.

Space-Charge Forces

Linear space-charge defocusing terms are calculated from the electric field components for a uniformly charged ellipse.[3] The space-charge (Coulomb repulsion) defocusing terms (constant term in the Mathieu equation) for nonrelativistic beams are

$$\Delta_L = JI\Psi f/(R_M^2 Z_M) \quad \text{(longitudinal)} \quad , \text{ and} \tag{1}$$

$$\Delta_T = JI\Psi(1 - f)/(R_M^2 Z_M) \quad \text{(transverse)} \quad , \tag{2}$$

where $J = 3Z_0 I/(4\pi)$, and f is a form factor that depends on $Z_m\sqrt{\Psi}/R_M$; that is, $f = f(Z_M\sqrt{\Psi}/R_M)$.

RFQ Parameters

The RFQ[4,5] is a device that provides transverse focusing, longitudinal sinusoidal bunching, and acceleration of beam particles. The RFQ's electrical properties are determined by using an electrostatic potential function, which then gives the shape of the vanes and is used to calculate the electric fields for beam-dynamics modeling.

We define the following parameters:[1] $k = 2\pi$, $A = (m^2 - 1)/[m^2 I_0(ka/\beta) + I_0(mka/\beta)]$, $\chi = [1 - AI_0(ka/\beta)]$, $B = V\chi/(a^2\gamma)$, and $\Delta_{rf} = \pi^2 VA \sin(\varphi_s)/2\gamma\beta_s^2$. Transverse particle motion in the RFQ is described by

$$d^2u/ds^2 + [\Delta_{rf} + B \sin(ks)]\, u = 0;\ u = (x \text{ or } Y) \ , \tag{3}$$

which is in the form of the Mathieu Equation. The transverse tune for particle motion averaged over one focusing period is obtained from Eq. (3), giving a wave number $\sigma_{To} = \sqrt{[\Delta_{rf} + B^2/(8\pi^2)]}$. The longitudinal motion is found by studying a particle's energy gain and phase change with respect to a "synchronous" particle through one focusing period. The longitudinal motion for small oscillations can be approximated by a harmonic oscillator having a longitudinal phase tune of $\sigma_{Lo} = \sqrt{-2\,\Delta_{rf}/\gamma^2}$.

The maximum electric field on the vane at quadrupole symmetry is $E_0 = V\sqrt{\chi}/a$. (Note that when designing an RFQ, the maximum electric field in a real

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device may not occur on the vane tip and for smooth vanes is typically 1.4 larger than given here.)[6]

We obtain from the equations in this section the two equations (4) and (5) and the flutter factor ψ (Eq. 6) used for the scaling laws

$$2\sigma_{To}^2 + \gamma^2\sigma_{Lo}^2 = E_o^2\chi/(4\pi^2\gamma^2a^2) \quad , \tag{4}$$

$$\sigma_{Lo}^2 = -\pi^2 E_o a\ A\ \sin(\varphi_s)/(\gamma^3\beta_s^2\sqrt{\chi}) \quad . \tag{5}$$

$$\psi = (2\pi + \sqrt{2\sigma_{To}^2 + \gamma^2\sigma_{Lo}^2})/(2\pi - \sqrt{2\sigma_{To}^2 + \gamma^2\sigma_{Lo}^2}) \quad . \tag{6}$$

(Note that ψ only depends on σ_{To} and σ_{Lo} for nonrelativistic beams.)

Combined RFQ and Space-Charge-Force Scaling Laws

We obtain relationships between the beam parameters I, ε_T, ε_L and the external-force-determined parameters σ_{To} and σ_{Lo}, which are related to the beam parameters through μ_T, μ_L. We then rearrange the RFQ equations so that the parameters in these equations reflect their dependence on the beam-dynamics quantities. Finally, we present these equations in a form that facilitates making RFQ designs for given, desired beam parameters.

We define "emittance" as the area/π enclosed by an ellipse having its semimajor axis (R_M, R'_M) or (Z_M, Z'_M) (obtaining $\varepsilon_T = R_M R'_M = \sigma_T R_M^2$, and $\varepsilon_L = Z_M Z'_M = \sigma_L Z_M^2$) and use the equipartitioning theorem[7,8] (which relates the free thermal energy in the longitudinal to the transverse direction) to combine these equations into $\varepsilon_T\sigma_T = \varepsilon_L\sigma_L$. We introduce a quantity g defined by $g = \sqrt{(\varepsilon_T\sigma_T)/(\varepsilon_L\sigma_L)}$, which indicates the deviation from equipartitioning. We can now show that $R_M = \sqrt{\varepsilon_T/\sigma_T}$, and $Z_M = \sqrt{(\varepsilon_T\sigma_T)}/(g\sigma_L)$. The form factor introduced above can now be written as

$$f(Z_M\sqrt{\psi}/R_M) = f[\sigma_T\sqrt{\psi}/(g\sigma_L)] \quad . \tag{7}$$

Combining Eqs. (1) and (2) and the equations in this section, we obtain two important relations

$$Jg(1-\mu_L)^{1/2}(1-\mu_T)^{1/4}/(\varepsilon_T^{3/2}\mu_L) = \sigma_{Lo}/(f\psi\sqrt{\sigma_{To}}) \text{ , and} \tag{8}$$

$$2f/(1-f) = \mu_L\sigma_{Lo}^2/(\mu_T\sigma_{To}^2) \quad . \tag{9}$$

We now study the RFQ Eqs. (4) and (5). We want to determine performance limits from the scaling laws. We therefore let $\varphi_s = -\pi/2$. The RFQ vane radius is related to the acceptance (A_T) by $a = \sqrt{A_T/\sigma_T}$. Let h be the ratio of acceptance/emittance ($h = A_T/\varepsilon_T$); then, $a = \sqrt{h}\ R_M = \sqrt{h\varepsilon_T/\sigma_T}$ In working with the beam-dynamics quantities, we constrained the longitudinal beam size using the equipartition theorem; therefore (for consistency), we do the same thing here. The bunch length typically is taken to be twice the synchronous phase (normalized to $\beta\lambda$) $Z_M = -\beta_s\varphi_s/(2\pi) = \beta_s/4$; therefore $\beta_s = 4\sqrt{\varepsilon_T\sigma_T}/(g\sigma_L)$.

This quantity defines a minimum $\beta(\beta_{min})$ because for $\beta > \beta_{min}$, we do not have to fill the entire longitudinal bucket. We can now rewrite the (ka/β) in the equations for A and X as $\pi\sqrt{h}g\sigma_L/(2\sigma_T)$. Combining equations we have

$$A = (m^2-1)/[m^2 I_o(\pi g\sqrt{h}\sigma_L/2\sigma_T) + I_o(m\pi g\sqrt{h}\sigma_L/2\sigma_T)] \quad ; \tag{10}$$

$$\chi = [1-A\ I_o(\pi g\sqrt{h}\sigma_L/2\sigma_T)] \quad ; \tag{11}$$

$$[E_o(1-\mu_T)^{1/4}/\sqrt{\varepsilon_T}]^2 = (2\sigma_{To}^2+\gamma^2\sigma_{Lo}^2)[4\pi^2\gamma^2h/(\chi\sigma_{To})] \quad ; \tag{12}$$

$$g^2(1-\mu_L)E_o/[(1-\mu_T)^{3/4}\sqrt{\varepsilon_T}] = 16\gamma^3\sqrt{\chi}\ \sigma_{To}^{3/2}/(\pi^2\sqrt{h}A) \quad . \tag{13}$$

Equations (7)-(9) and (10)-(13) contain all the information needed to obtain our scaling laws. We will study two cases. The first case assumes equipartitioning (g = 1) and equal transverse and longitudinal tune depressions ($\mu_L = \mu_T = \mu$).

In the second scaling case, we let the ratio μ_L/μ_T be any fixed quantity, but we require that

$$g = \sqrt{(1-\mu_T)/(1-\mu_L)} = \sqrt{\varepsilon_T\sigma_T/\varepsilon_L\sigma_L} \quad . \tag{14}$$

(Case 1 is a special case of two. We write the equations for both cases.) With this restriction, Eqs. (7)-(9) and Eqs. (10)-(13) become the equations listed below. [Equation (15) defines L_1, as $J(1-\mu_T)^{3/4}/(\varepsilon_T^{3/2}\ \mu_L)$.]

BEAM-DYNAMICS EQUATIONS:

$$f = f(\sigma_{To}\sqrt{\psi}/\sigma_{Lo}) \quad , \tag{15}$$

$$L_1 = J(1-\mu_T)^{3/4}/(\varepsilon_T^{3/2}\mu_L) = \sigma_{Lo}/(f\psi\sqrt{\sigma_{To}}) \quad , \tag{16}$$

$$2f/(1-f) = \mu_L\sigma_{Lo}^2/(\mu_T\sigma_{To}^2) \quad . \tag{17}$$

RFQ EQUATIONS:

$$A = (m^2-1)/[m^2 I_o(\pi\sqrt{h}\sigma_{Lo}/2\sigma_{To}) + I_o(m\pi\sqrt{h}\sigma_{Lo}/2\sigma_{To})] \quad , \tag{18}$$

$$\chi = [1 - A\ I_o(\pi\sqrt{h}\sigma_{Lo}/2\sigma_{To})] \quad , \tag{19}$$

$$[E_o(1-\mu_T)^{1/4}/\sqrt{\varepsilon_T}]^2 = (2\sigma_{To}^2 + \gamma^2\sigma_{Lo}^2)[4\pi^2\gamma^2h/(\chi\sigma_{To})] \quad , \tag{20}$$

$$[E_o(1-\mu_T)^{1/4}/\sqrt{\varepsilon_T}] = 16\gamma^3\sqrt{\chi}\ \sigma_{To}^{3/2}/(\pi^2\sqrt{h}A) \quad . \tag{21}$$

Equations (6), (15), and (17) uniquely determine σ_{Lo} versus σ_{To} for a fixed μ_L/μ_T ratio. (We have assumed that $\gamma = 1$, a good approximation for the ion beams of interest.) We can treat $[E_o(1-\mu_T)^{1/4}/\sqrt{\varepsilon_T}]$ in Eqs. (20) and (21) as a single entity: $L_2 = (E_o(1-\mu_T)^{1/4}/\sqrt{\varepsilon_T})$. The right-hand sides of Eqs. (20) and (21) depend only on σ_{To}, σ_{Lo}, and m. We can solve these equations in a self-consistent manner to eliminate the modulation (m) dependence. Because there is a unique relationship between σ_{To} and σ_{Lo}, the quantity L_2 is a unique function of σ_{To}. Similar considerations show that the right-hand side of Eq. (16) depends only on σ_{To}. We therefore have a parametric relationship between the quantity L_2 and the left-hand quantity of Eq. (16) (L_1). In Fig. 1, we plot $(3Z_oL_2)/(4\pi L_1) = L_3$ versus $(4\pi L_1/(3Zo) = L_4$ for $\mu_L = \mu_T$. We revert to SI units in making this plot so that the scaling will be more obvious. The units used are E_o^{SI} (volts/meter), ε_T^{SI} (meter-radians), I^{SI} (amperes), M_o (electron volts). (The dimensionless parameters E_o, ε_T, A_T, and I in L_3 and L_4 are the following functions of the dimensioned parameters: $Eo = Eo^{SI}\lambda/Mo$, $\varepsilon_T = \varepsilon_T^{SI}/\lambda$, $A_T = A_T^{SI}/\lambda$, and $I = I^{SI}/Mo$.)

It is remarkable that the two constraints of equipartitioning and $\mu_L = \mu_T$ have led to a scaling relation defined as a single curve. Given a set of requirements on particle type, beam current, emittance, and maximum acceptable space charge μ, only the peak surface vane-tip electric field at quadrupole symmetry and rf wavelength remain to be adjusted (within the

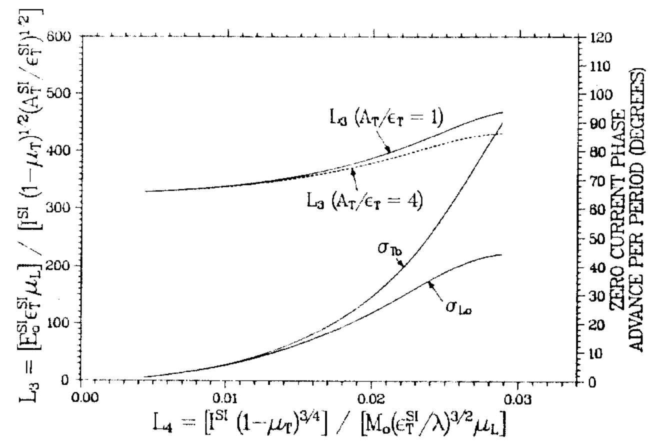

Fig. 1. Scaling law curves ($\mu_L = \mu_T$).

constraints of the scaling curve) to determine the maximum performance capability of a given RFQ design. One other scaling relation is indicated in Fig. 1 when the acceptance A_T is not equal to the emittance ϵ_T. We plot L_3 versus L_4 for $A_T/\epsilon_T = 4$, where L_3 is divided by $\sqrt{A_T/\epsilon_T}$. The minor difference in the curves indicates that we can design for $A_T = \epsilon_T$, then multiply the resulting electric field by $\sqrt{A_T/\epsilon_T}$.

Almost the same considerations apply to Case 2 as to Case 1 above, except that Eq. (17) now has a μ_L/μ_T ratio dependence. We will therefore get a different but unique scaling curve for each different ratio of μ_L/μ_T. Two cases are shown in Fig. 2. For reference purposes in Figs. 1 and 2, we show curves of σ_{Lo} and σ_{To} versus L_4.

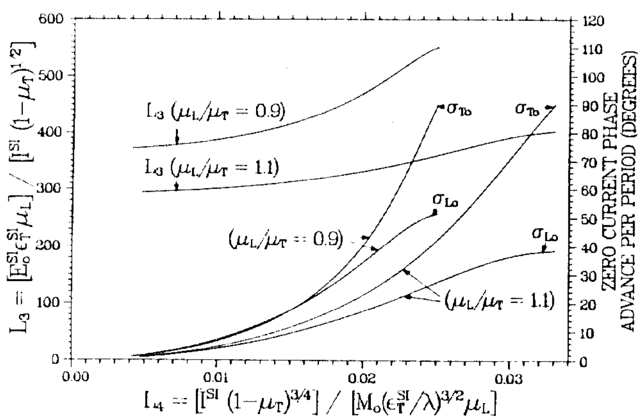

Fig. 2. Scaling law curves ($\mu_L \neq \mu_T$).

RFQ Design Examples and Discussion

Several observations concerning Fig. 1 also apply to Fig. 2 when the ratio μ_L/μ_T is fixed and g is constrained as in Eq. (14). First, any RFQ design that satisfies equipartitioning and $\mu_L = \mu_T$ will lie on the curve L_3 versus L_4. Second, suppose we wish to maximize beam current for a given zero current tune without regard to emittance. We can eliminate ϵ_T^{SI} between L_4 and L_3 and solve for I^{SI} to obtain

$$I^{SI} = \lambda^3 (E_0^{SI})^3 \mu_L/(L_4^2 M_0^2 L_3^3) \propto \epsilon_T^{SI} \propto \lambda^3 \ (L_3, L_4, E_0 \text{ fixed}).$$

We can increase current for a given beam-dynamics characterization by increasing the rf wavelength. Note that with L_3 fixed, the emittance will increase linearly with current.

If the desired goal is to maximize the beam brightness for a given zero current tune, we can write $I^{SI}/(\epsilon_T^{SI})^2 = M_0^2 L_4^2 L_3 \ \mu_L/[E_0^{SI}(1-\mu_T)\lambda^3] \propto 1/\lambda^3$. In this case, the beam brightness will increase with decreasing rf wavelength. For a given beam energy, power is proportional to current; therefore the increase in beam brightness will be accompanied by a proportional decrease in beam power. (Current and emittance are proportional to λ^3.)

As a simple example of using Fig. 1, suppose we wish to design a RFQ as an injector to an existing drift-tube linear accelerator. The existing device has an rf frequency of 425 MHz and accelerates a proton beam having 0.2-A current, 2×10^{-6} π•m•rad transverse normalized emittance, and a maximum space charge μ of 0.7. We find

$$L_4 = I^{SI}(1 - \mu)^{3/4} \lambda^{3/2}/[M_0(\epsilon_T^{SI})^{3/2}\mu] = 0.0258.$$

From Fig. 1 we find $L_3 = 445$. We then calculate E_0 using L_3 giving 34.8×10^6 V/m. This electric field is almost twice the Kilpatrik field limit and does not include any safety factors. If the beam-current requirement for the above case is 0.1 A, then the electric field determined from Fig. 1 is 13.7×10^6 V/m.

Conclusion

We have derived several scaling relations to show the various tradeoffs involved in choosing RFQ designs and have provided curves to help choose starting points in parameter space for optimizing an RFQ for a particular requirement. We have shown that there is a unique scaling curve that relates the beam current, emittance, and particle mass with RFQ frequency and maximum vane-tip electric field and with space-charge tune depression--if we assume equipartitioning and equal longitudinal and transverse tune depressions. Finally, we have presented several examples for designing RFQs using our procedure.

Acknowledgments

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