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ROBINSON INSTABILITY AND BEAM LOADING*

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Abstract

The Robinson instability problem¹ is developed in three stages. The first step is to derive the synchrotron oscillation equations in the abscence of beam loading (unloaded case). Next, the equations are evaluated in the presence of beam loading at the fundamental rf frequency (statically loaded case). Finally, the system is redeveloped taking into account beam loading at the synchrotron sidebands (dynamically loaded case).

Following the theoretical development, the results are applied to calculate the synchrotron frequency in the presence of beam-loading, automatic-gain-control, and automatic-tune-control. The results of this calculation are compared with data from the NSLS VUVring, a 750 MeV electron storage ring.

Unloaded Case

In electron storage rings, radio-frequency accelerator cavities are necessary to replenish the energy continually lost by the orbiting electrons to synchrotron radiation. The time-dependence of the electric field in the cavity couples with the energydependence of the electron revolution period to produce 'synchrotron oscillations.'

Consider a circulating electron with an energy that deviates by an amount ϵ from the design energy, E_o . Corresponding to this energy deviation, there is a deviation, $\delta\tau$, of the rotation period from the synchronous period, τ_o , given by

$$\delta \tau = \frac{d\tau}{dE} \epsilon$$
 (1)

Expanding the derivative of τ in terms of partials with respect to the length of the orbit, L, and the velocity of the particle, v, yields

$$\delta \tau = \left(\frac{\partial \tau}{\partial L} \frac{dL}{dE} + \frac{\partial \tau}{\partial v} \frac{dv}{dE} \right) \epsilon .$$
 (2)

Introducing the definition of the 'momentum compaction factor',

$$n \equiv \frac{E_{o}}{\tau_{o}} \left(\frac{\partial \tau}{\partial L} \frac{dL}{dE} + \frac{\partial \tau}{\partial v} \frac{dv}{dE} \right)$$
(3)

gives

$$\delta \tau = \frac{n}{E_o} \varepsilon$$
 (4)

In general, due to relativistic effects, η experiences a change of sign at the 'transition energy', E_t . In electron machines though, injection typically occurs above transition so that η , as defined above, is strictly positive.

A more convient variable than $\delta \tau$ is Φ , the phase of the rf voltage in the accelerating cavity when the electron arrives. The change in Φ per revolution is related to $\delta \tau$ by

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$$\Delta \Phi = \omega_{\rm rf} \, \delta \tau \tag{5}$$

where ω_{rf} is the frequency of the accelerating voltage. Then, introducing the 'smooth approximation',

$$\Phi \equiv \Delta \Phi / \tau ,$$
(6)

yields the 'phase equation',

$$\dot{\Phi} = \frac{\omega_{\rm rf}}{E_{\rm o}} \varepsilon \,. \tag{7}$$

Considering now the change in energy of a circulating electron per revolution, $\Delta\epsilon_{\rm e}$ one has

$$\Delta \varepsilon = e \hat{V}_{c} \cos \Phi - U(E) , \qquad (8)$$

where e is the absolute value of the electron charge, $\hat{\mathbb{V}}_{_{C}}$ is the amplitude of the cavity voltage and U(E) is the energy lost to radiation per revolution. Writing Φ as the sum of a time-independent part, $\phi_{_{\rm S}}$ and a small time-dependent part, $\phi_{_{\rm S}}$ and expanding U about E_ gives

$$\Delta \varepsilon = e \hat{V}_{c} \left(\cos \phi_{s} - \phi \sin \phi_{s} \right) - U_{o} - \frac{dU}{dE} \varepsilon .$$
 (9)

The last term on the right-hand side leads to 'synchrotron damping', a seperate topic, and will therefore be dropped.

Using the time-independent phase, $\phi_{\rm s}$, to cancel U $_o$ yields the defining equation for the synchronus phase.

$$\phi_{s} = a\cos\left(\frac{U_{o}}{e\hat{V}_{c}}\right)$$
(10)

The remaining terms in the equation for $\Delta\epsilon,$ yield, after smoothing, the 'energy equation.'

$$\dot{\epsilon} = -\frac{e\hat{v}_c}{\tau}\sin\phi_s \star \phi \tag{11}$$

Differientiating the phase equation and substituting into it the energy equation gives the 'oscillation equation'.

$$\ddot{\phi} = -\frac{\omega_{\rm rf} n}{E_{\rm o}} \frac{e \hat{V}_{\rm c}}{\tau_{\rm o}} \sin \phi_{\rm s} \star \phi \qquad (12)$$

Defining the synchrotron frequency, ω_s as

$$\omega_{\rm s}^2 \equiv \frac{\omega_{\rm rf} n}{E_{\rm o}} \frac{eV_{\rm c}}{\tau_{\rm o}} \sin\phi_{\rm s} , \qquad (13)$$

the oscillation equation can be rewritten as

 $\ddot{\phi} + \omega_{\rm s}^2 \phi = 0 . \qquad (14)$

Since n is poistive, stability of the oscillations requires that

$$\sin(\omega_{c}) > 0 \text{ or } 0 < \phi_{c} < \pi.$$
 (15)

But from eq. 10^{7} , since U₀ is positive, one has

$$-\pi/2 < \phi_0 < \pi/2.$$
 (16)

Consequently,

$$0 < \phi_{a} < \pi/2. \tag{17}$$

In the absence of beam-loading, the cavity voltage, \overline{V}_c , is equal to that supplied by the rf generator, \overline{V}_g . The situation is depicted in the phasor diagram, fig. 1. The reference vector chosen for this diagram is $-\overline{I}_b$, the negative of the beam current or the location of the electrons. Thus the electrons are seen to ride the falling side of the rf wave.



Figure 1. Phasor diagram of cavity voltage, unloaded case.



Figure 2. Phasor diagram of cavity voltage, statically loaded case.

Statically Loaded Case

This case is illustrated in fig. 2. Taking Z_0 to be the impedance through which the beam current contributes to the fields in the accelerator cavity, the phase of Z_0 is defined as

$$\phi_{zo} \equiv \operatorname{Im}(\ln(Z_{z})). \tag{18}$$

Soley for the purpose of drawing fig. 2, it has been assumed that

$$-\pi/2 < \phi_{zo} < 0.$$
 (19)

The voltage in the cavity is given by

$$\tilde{\tilde{v}}_{c} = \left[\hat{\tilde{v}}_{g} \cos \phi_{g} - \hat{I}_{b} \middle| Z_{o} \middle| \cos \phi_{zo} \right] \cos(\omega_{rf}t)$$
(20)
$$- \left[\hat{\tilde{v}}_{g} \sin \phi_{g} - \hat{I}_{b} \middle| Z_{o} \middle| \sin \phi_{zo} \right] \sin(\omega_{rf}t) ,$$

where \hat{I}_b is the Fourier amplitude of the beam current at ω_{rf} . The accelerating voltage seen by the beam is

Equating the ϕ independent term with the energy loss term gives

$$\phi_{g} = \arccos\left(\frac{U_{o}/e + \hat{I}_{b} | Z_{o} | \cos \phi_{z0}}{\hat{V}_{g}}\right); \ \phi_{c} = \arccos\left(\frac{U_{o}}{e\hat{V}_{c}}\right).$$
(22)

Solving for the synchrotron frequency gives

$$\omega_{s}^{2} = \frac{\omega_{rf}^{ne}}{\frac{E_{o}\tau_{o}}{E_{o}\tau_{o}}} \left(\hat{V}_{g} \sin\phi_{g} - \hat{I}_{b} | Z_{o} | \sin\phi_{zo} \right)$$
(23)
$$= \frac{\omega_{rf}^{ne}}{\frac{E_{o}\tau_{o}}{E_{o}\tau_{o}}} \hat{V}_{c} \sin\phi_{c}.$$

Dynamically Loaded Case

In the preceding section, a calculation was performed to determine the frequency of synchrotron oscillations in the presence of beam loading. The calculation was not self-consistent in that the beam current used to load the cavity did not itself contain the synchrotron oscillations. The appropriate beam current to use is

$$\tilde{I}_{b} = R_{e} \left[e^{i \left(\omega_{rf} t + \phi \right)} \right]$$
(24)

with .

$$\phi = \alpha e^{\beta t} \cos(\omega_{s} t)$$
 (25)

where α , the amplitude of the phase oscillation, is an infintesimal and β is a growth rate to be determined along with ω_g . The voltage in the cavity is

$$\tilde{\mathbf{V}}_{c} = \left\{ \hat{\mathbf{V}}_{g} \cos \phi_{g} - \hat{\mathbf{I}}_{b} \left[\left| \mathbf{Z}_{o} \right| \cos \phi_{zo} - \frac{\alpha e^{\beta t}}{2} \left(\left| \mathbf{Z}_{+} \right| \sin(\omega_{s} t + \phi_{z+}) \right. \right. \right. \right.$$

$$\left. + \left| \mathbf{Z}_{-} \right| \sin(-\omega_{s} t + \phi_{z-}) \right] \right\} \cos(w_{rf} t)$$

$$\left. - \left\{ \hat{\mathbf{V}}_{g} \sin \phi_{g} - \hat{\mathbf{I}}_{b} \left[\left| \mathbf{Z}_{o} \right| \sin \phi_{zo} - \frac{\alpha e^{\beta t}}{2} \left(\left| \mathbf{Z}_{+} \right| \cos(\omega_{s} t + \phi_{z+}) \right. \right. \right. \right. \right.$$

$$\left. + \left| \mathbf{Z}_{-} \right| \cos(-\omega_{s} t + \phi_{z-}) \right] \right\} \sin(w_{rf} t)$$

where Z_+ and Z_- are the cavity impedance at the upper and lower sidebands respectively. The accelerating voltage seen by the beam is

$$V_{c} = \hat{V}_{g} \cos \phi_{g} - \hat{I}_{b} |Z_{o}| \cos \phi_{zo} - \hat{I}_{b} - \frac{\alpha e^{\beta t}}{2} (|Z_{+}| \sin(\omega_{s} t + \phi_{z+}) - (27) + |Z_{-}| \sin(\omega_{s} t + \phi_{zt})) - (\hat{V}_{g} \sin \phi_{g} - \hat{I}_{b} |Z_{o}| \sin \phi_{zo}) \frac{\beta t}{2}$$

Substituting into the energy equation and cancelling the phase-oscillation-independent terms with the energy-loss term results in same equation for the stable phase angle as in the statically loaded case. The oscillation equation resulting from the phasedependent terms is

$$\vec{\phi} = \frac{\omega_{rf}^{ne}}{E_{o}\tau_{o}} \left[\left(-V_{g}^{\sin\phi} + \hat{I}_{b}^{m} Im(Z_{o}^{m} - \bar{Z}^{m}) \right) \cos(\omega_{s}^{m}t) - \hat{I}_{b}^{m} Re(\Delta Z) \sin(\omega_{s}^{m}t) \right] \alpha e^{\beta t}$$
(28)

where

$$\overline{Z} = \frac{1}{2} (Z_{+} + Z_{-}) ; \quad \Delta Z = \frac{1}{2} (Z_{+} - Z_{-}) .$$
 (29)

Expanding $\ddot{\phi}$ using eq. 25 and taking advantage of the orthogonality of sine and cosine, the oscillator equation can be decomposed to yield the system of equations

$$\beta^{2} - \omega_{s}^{2} = \frac{\omega_{rf}^{ne}}{E_{o}\tau_{o}} \left(-V_{g} \sin\phi_{g} + \hat{I}_{b} Im(Z_{o} - \bar{Z}) \right)$$
(30)

and

$$2\beta\omega_{\rm s} = \frac{\omega_{\rm rf}^{\rm ne}}{E_{\rm o}^{\rm \tau} c_{\rm o}} \quad \hat{\rm I}_{\rm b} {\rm Re}(\Delta Z) \quad . \tag{31}$$

At this point, it is necessary to fix the sign of ω_{s} . Since ϕ is the phase by which the cavity voltage leads the electrons, it is clear that when ϕ is maximally positive, ϵ must be negative going and thus w_{s} must be positive. The first stability condition then follows directly from eq. (31) by requiring β , the growth-rate, to be non-positive. Thus, stability against phase-oscillations requires

$$\operatorname{Re}(Z_{+}) > \operatorname{Re}(Z_{+}) . \tag{32}$$

Comparing the right hand sides of equations 23 and 30, one finds that the beam loading term proportional to $\operatorname{Im}(Z_O)$ in eq. 23 is countered in eq. 30 by the term in $\operatorname{Im}(\overline{Z})$. The term in Z_O is due to infitesimal deviations in the arrival time of the electrons whereas the term in \overline{Z} is due to infinitesimal currents in the side-bands. Substituting for β and using \hat{V}_C to represent the amplitude of the cavity voltage due to non-infinitesimal current sources, eq. 30 can be rewritten as

$$\omega_{s}^{2} - \frac{1}{\omega_{s}^{2}} \left(\frac{\omega_{rf} \eta e}{E_{o} \tau_{o}} \frac{\hat{I}_{b}}{2} \operatorname{Re}(\Delta Z) \right)^{2} = \frac{\omega_{rf} \eta e}{E_{o} \tau_{o}} \left(\hat{v}_{c} \sin \phi_{s} + \hat{I}_{b} \operatorname{Im}(\bar{Z}) \right) \quad (33)$$

or using eq. 23 as

$$\frac{\omega_{s}^{2}}{\omega_{s0}^{2}} - \frac{\omega_{s0}^{2}}{\omega_{s}^{2}} \left(\frac{\hat{\mathbf{I}}_{b}}{2\hat{\mathbf{V}}_{c}\sin\phi_{s}} \operatorname{Re}(\Delta Z) \right)^{2} = 1 + \frac{\hat{\mathbf{I}}_{b}\operatorname{Im}(\bar{Z})}{\hat{\mathbf{V}}_{c}\sin\phi_{s}} \quad . \tag{34}$$

Neglecting the second term on the left hand side of eq. 34 leads to an underestimate of the synchrotron frequency. Consequently, calculation of the beam current required to shift the synchrotron frequency to zero, neglecting this term, yields a conservative estimate of the maximum allowable current. This results in the second of Robinson's stability criteria.

$$Im(Z_{o}) > - \frac{\hat{V}_{c} \sin \phi_{g}}{\hat{I}_{b}}$$
(35)

If one now assumes the single-pole resonator form for the cavity impedance,

$$Z(\omega) = \frac{R}{1+iQ(\frac{\omega}{\omega} - \frac{\omega}{\omega})}$$
(36)

where R is the shunt impedance, Q is the quality factor, and $\omega_{\rm C}$ is the resonant frequency, then results 32 and 35 can be written as

$$0 < - \operatorname{Im}(Z_{o}) < \frac{\hat{v}_{c} \sin\phi_{s}}{\hat{i}_{b}} \quad . \tag{37}$$

Beam-Loading in the VUV-Ring

The rf system of the NSLS VUV-ring contains two slow feed-back systems. The first is the automaticgain-control circuit, AGC, which maintains constant the total rf voltage in the accelerator cavity. The other is the automatic-tune-control circuit, auto-tune, which adjusts various tuning mechanisms to maintain constant the phase relationship between the rf generator current and total voltage.

The affect of AGC has already been incorporated in the preceeding derivation simply by using $V_{\rm C}$ in equations 33 and 34. The affect of auto-tune is to make $\omega_{\rm C}$ in equation 36 a function of $I_{\rm b}$. With the above somewhat simplified, assumptions concerning auto-tune, the functional dependence of $\omega_{\rm C}$ is given by

 $\omega_{c} = \frac{\omega_{rf}}{2} \left[-x + (x^{2} + 4)^{1/2} \right]$

where

$$X \equiv \left[\left((1 - r^{-2})^{1/2} + tan\delta_{o} \right) \frac{\hat{I}_{b}R}{r} \left(\frac{U_{o}}{e} \right)^{-1} + tan\delta_{o} \right] / Q, \quad (39)$$

$$\mathbf{r} \equiv \frac{\mathbf{e} \hat{\mathbf{V}}_{\mathbf{c}}}{\mathbf{U}_{\mathbf{o}}} , \qquad (40)$$

and δ_0 is the zero-current turning angle.

A computer program has been written to solve the system of equations (34, 36, 38) for $\omega_{\rm g}$. Figure 3 compares experimental data from the NSLS VUV-Ring with results of this calculation. In the calculation, $U_{\rm g}$ = 14.6 KeV, $\delta_{\rm g}$ = 10°, and R/Q = 65 Ω . The theoretical curves, in order of increasing $\omega_{\rm g}$, correspond to Q = 4000, 8000 and 12000.



Figure 3. Comparison of data with calculation (see text).

Clearly, there is little agreement between data and calculation other than the sign of the slope. Great uncertainty exists in the interpretation of the data as well as values of constants in the calculation. Also no estimate of the affects of higher-order-modes has been made.

Further experimental measurements are planned including the region of parameter space where $\omega_{\rm S}$ decreases with increasing beam current.

Reference

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(38)