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SELF-CONSISTENT NON-K-V DISTRIBUTIONS FOR PERIODIC FOCUSING SYSTEMS

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Summary

The method to construct a self-consistent particle phase space distribution, representing a continuous charged particle beam moving through a spatially constant focusing device, is reviewed. Subsequently, an approach to extend this theory to periodic focusing systems is outlined and discussed.

1. Introduction

It is known[1,2] that for constant focusing forces any particle distribution, f, in the 4-dimensional transverse phase space depending only on the Hamiltonian of the system, $f = f(H)$, is a stationary (self-consistent) solution of Vlasov's equation. For periodic systems under space charge conditions, a self-consistent distribution is obtained if all phase space points form a homogeneously populated hyperellipsoidal surface. Since this type of distribution, usually designated as the "Kapchinskij-Vladimirskij" (K-V) distribution, is not "physically realistic", it is important for our understanding of beam transport phenomena to ascertain whether self-consistent non-K-V phase space distributions exist for periodic focusing systems.

In this report, a canonical transformation will be presented, that correlates the beam transport in a constant focusing system with the beam transport in an "equivalent" periodic focusing system. If the distribution in the constant focusing system is self-consistent, then, as a consequence of the correlation, the distribution in the periodic focusing system will also be self-consistent. The conditions for this mapping in order to be valid will be investigated. Finally, results of computer simulations will be presented to verify this approach.

2. Constant Focusing Systems

If we assume the beam to be a nonneutral, collisionless plasma, Liouville's theorem is valid. Considering an unbunched beam moving through a spatially constant focusing system, an infinite variety of stationary particle phase space distributions exists, even if the space charge forces are not negligible. Taking H as the Hamiltonian of a single particle and the axial position s as the independent variable instead of the time, the condition for a phase space distribution f to be self-consistent is obtained readily from Liouville's theorem:

$$\frac{df}{ds} = 0 \quad \Longleftrightarrow \quad \frac{\partial f}{\partial s} + \{H,f\} = 0 \quad , \qquad (1)$$

$$f = f(H) \quad \Rightarrow \quad \frac{\partial f}{\partial s} = 0 \quad .$$

For a round beam in an azimuthally symmetric focusing system, the Hamiltonian can be written in the following dimensionless form:

$$H(r,r') = \tfrac{1}{2}r'^2 + \tfrac{1}{2}k_0^2 r^2 + \tfrac{1}{2}K\cdot V(r) \quad , \qquad (2)$$

where $r^2 := x^2 + y^2$ and $r'^2 := x'^2 + y'^2$ denote the radial position and the transverse angle, respectively. The "generalized perveance" K is given by:

$$K = 2I/I_0\beta^3\gamma^3 \quad , \quad I_0 = 4\pi\varepsilon_0 c^3/q \quad .$$

The focusing constant k_0 is correlated with the strength of the focusing field. If we use a long solenoid as the focusing device, then for a charged particle of the charge state q, rest mass m_0 and velocity $c\beta$, k_0 is given by $k_0 = qB_0/2m_0c\beta\gamma$. For this focusing system, the coordinates r,r' refer to the rotating Lamor frame. The dimensionless collective space charge potential, $V(r)$, produced by all beam particles follows from Poisson's equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}V(r)\right) = -4\pi\cdot g(r) \quad , \qquad (3)$$

wherein $g(r)$ describes the charge density in the transverse real space. It is obtained by integrating the phase space distribution function over all transverse angles:

$$g(r) = \pi\cdot\int_0^{a'^2(r)} f(H(r,r'))\, d(r'^2) \quad . \qquad (4)$$

The upper boundary of integration $a'^2(r) := r'^2_{max}(r)$ follows from the maximum transverse energy H_0 of a beam particle:

$$a'^2(r) = k_0^2\cdot(a^2 - r^2) + K\cdot(V(a) - V(r)) \quad ,$$

where "a" stands for the beam radius. If we introduce the quantity "effective potential" $W(r)$ as the sum of the focusing and the space charge potential:

$$W(r) = \tfrac{1}{2}k_0^2 r^2 + \tfrac{1}{2}K\cdot V(r) \quad ,$$

equations (3) and (4) can be combined yielding the following inhomogeneous integro-differential equation for the self-consistent effective potential $W(r)$ (i.e. the effective potential of a stationary phase space distribution):

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}W(r)\right) - 4\pi^2 K\cdot\int_{W(a)}^{W(r)} f(H)\, dH = 2k_0^2 \quad , \qquad (5)$$

wherein $f(H)$ is a normalized arbitrary phase space density function. Equation (5) has analytical solutions at least for two types of functions $f(H)$.

For the "K-V" distribution function:

a) $$f(H) = c_0\cdot\delta(H - H_0) \quad ,$$

where δ stands for the Dirac-δ-function, the solution of (5) is given by:

$$W(r) = \tfrac{1}{2}k^2 r^2 + \tfrac{1}{2}K \quad , \quad k^2 := k_0^2 - K/a^2 \quad .$$

Due to the quadratic effective potential in absence as well as in presence of space charge forces, this type of distribution leads to linear equations of motions throughout.

b) $$f(H) = c_1\cdot\Theta(H_0 - H) \quad ,$$

where Θ denotes the step function, means to fill the phase space homogeneously inside a bounded volume (which remains constant according to Liouville's theorem). Therefore, $f(H)$ is usually designated as the "water bag" distribution, referring to the analogy of an incompressible fluid inside a closed bag, which can change its shape but not its volume. The solution function $W(r)$ of equation (5) can be expressed in terms of the modified Bessel function I_0:

$$W(r) = W(a)\cdot\left[1 - \frac{4}{\kappa^2 a^2}\cdot(1 - I_0(\kappa r)/I_0(\kappa a))\right] \quad .$$

It contains a dimensionless space charge parameter κa, that can be defined only implicitly:

$$K/k_0^2 a^2 = I_2(\kappa a)/I_0(\kappa a) \quad .$$

$W(r)$ is quadratic only in the zero current limit ($\kappa a \to 0$). For high currents it is nearly constant in the interior

of the beam and rises sharply at its boundary. Therefore, it is often called a "reflecting wall" potential.
For the Gaussian distribution:

c) $$f(H) = c_2 \cdot \exp(-c_3 H/H_0)$$

equation (5) leads to a differential equation, which can only be solved numerically. Especially for high currents, $W(r)$ does not differ very much from the equivalent "waterbag" case.

3. Periodic Focusing Systems

For economical reasons, realistic focusing channels consist of discrete focusing lenses. Their maximum acceptance is obtained, if these lenses (i.e. quadrupoles or solenoids) are placed periodically along the beam transport line. In the following, $\phi(\xi,\eta,\xi',\eta')$ will denote a stationary phase space distribution function in a constant focusing system, and $f(x,y,x',y';s)$ a distribution in a periodic system. The length of one focusing period will be denoted by "S".
As before, an unbunched beam for which Liouville's theorem (1) is valid will be considered. By combining (1) with the equation of motion for the individual particle, we obtain Vlasov's equation:

$$\frac{\partial f}{\partial s} + \sum_i \left[x_i' \frac{\partial f}{\partial x_i} - \left(k_i^2(s)\cdot x_i - \tfrac{1}{2}K\cdot E_i\right)\frac{\partial f}{\partial x_i'}\right] = 0 \ . \quad (6)$$

The electric field components E_x, E_y included herein are correlated with the particle distribution function, f, by Poisson's equation:

$$\operatorname{div} \vec{E}(x,y;s) = 4\pi \cdot \iint_{-\infty}^{\infty} f(x,y,x',y';s)\, dx'dy' \ .$$

Obviously, $f(x,y,x',y';s)$ is a stationary solution of Vlasov's equation[3], if it can be expressed as a function of a constant of motion C : $f = f(C)$. In this case, beam transport is a reversible process and therefore associated with a conservation of the beam entropy.
For constant focusing systems, the Hamiltonian (2) is a constant of motion ($C = H$), so any particle distribution depending only on the Hamiltonian of the system is a stationary solution of Vlasov's equation, as already stated in section 2.
For periodic channels, the appropriate Hamiltonian is not a constant of motion, hence $f = f(H)$ is no longer a sufficient condition for the distribution function f to be self-consistent. In order to establish the constant of motion for a beam particle moving in a periodic channel, we map the s-dependent Hamiltonian of the periodic system via an s-dependent canonical transformation to a new constant Hamiltonian. Physically, this means to replace the periodic beam transport channel by an "equivalent" constant focusing system. The "smooth approximation" technique[4], that deals with the averaged values of periodically modulated beam envelopes in order to obtain analytical formulas describing the averaged beam behavior, is based implicitly upon this replacement.
The canonical transformation can be written clearly, if we introduce the quantity "generalized phase advance"[5]. With the rms quantities $\langle\varepsilon_x\rangle$ and $\langle x^2\rangle$, it can be defined as:

$$\psi_x(s) = \int_{s_0}^{s} \langle\varepsilon_x\rangle/\langle x^2\rangle(z)\, dz \ ,$$

and similarly for the y-direction. The difference of the generalized phase advances of the periodic (p) and the constant (c) transport system will be abbreviated by $\chi(s)$:

$$\chi(s) = \psi_{x,p}(s) - \psi_{x,c}(s) \ .$$

The transformation that correlates the particles forming the distributions $f = f(x,y,x',y';s)$ in the periodic and $\phi = \phi(\xi,\eta,\xi',\eta')$ in the constant transport system is given by[6]:

$$\begin{bmatrix} x \\ x' \end{bmatrix} = B\ A \begin{bmatrix} \xi \\ \xi' \end{bmatrix} \ , \quad (7)$$

wherein

$$A = \begin{bmatrix} \cos\chi & \sqrt{\langle\xi^2\rangle/\langle\xi'^2\rangle}\cdot\sin\chi \\ -\sqrt{\langle\xi'^2\rangle/\langle\xi^2\rangle}\cdot\sin\chi & \cos\chi \end{bmatrix}$$

$$B = \begin{bmatrix} \sqrt{\langle x^2\rangle/\langle\xi^2\rangle} & 0 \\ \langle xx'\rangle/\sqrt{\langle x^2\rangle\langle\xi^2\rangle} & \sqrt{\langle\xi^2\rangle/\langle x^2\rangle} \end{bmatrix} .$$

The corresponding transformation must be used for the y,y'-plane. According to the unit determinants of the matrices A and B, the beam emittances pertaining to the distributions f and ϕ agree.
With $U(x,y;s)$ as the space charge potential of the new distribution f, and under the precondition that the function $r(x,y;s)$, defined as:

$$r(x,y;s) = \tfrac{1}{2}K\left[U(x,y;s) - \tfrac{1}{2}x^2\langle x\frac{\partial U}{\partial x}\rangle/\langle x^2\rangle - \tfrac{1}{2}y^2\langle y\frac{\partial U}{\partial y}\rangle/\langle y^2\rangle\right]$$

is an invariant with regard to the transformation (7), we obtain a new Hamiltonian, that has the same form as the original one (2):

$$H = \tfrac{1}{2}(x'^2 + y'^2) + \tfrac{1}{2}(k_x^2(s)x^2 + k_y^2(s)y^2) + \tfrac{1}{2}K\cdot U(x,y;s).$$

As a consequence of the canonical transformation, the function $k_x(s)$ is defined by the following differential equation:

$$\frac{d^2}{ds^2}\sqrt{\langle x^2\rangle} + k_x^2(s)\sqrt{\langle x^2\rangle} - \tfrac{1}{2}K\langle xE_x\rangle/\sqrt{\langle x^2\rangle} - \langle\varepsilon_x\rangle^2/\sqrt{\langle x^2\rangle^3} = 0 \ .$$

The corresponding equation holds for the y-direction. These coupled differential equations are known as the "rms envelope equations"[7]. The emittance $\langle\varepsilon_x\rangle$ is correlated by eq. (7) with the emittance of the distribution ϕ in the equivalent constant focusing system. If ϕ is a stationary distribution, hence associated with a constant emittance, the emittance of the transformed distribution will be constant, too. In this case, the rms envelope equations are closed and can therefore be used to determine the S-periodic moments $\langle x^2\rangle$, $\langle y^2\rangle$, $\langle xx'\rangle$ and $\langle yy'\rangle$ pertaining to a matched beam at a given position s within the focusing period, which are included in matrix B.
The constant of motion C for a particle moving through a periodic system is obtained by expressing the Hamiltonian of the equivalent constant focusing system in the variables of the periodic:

$$C = \tfrac{1}{2}(x^2\langle x'^2\rangle - 2xx'\langle xx'\rangle + x'^2\langle x^2\rangle)/\langle\xi^2\rangle + \tfrac{1}{2}(y^2\langle y'^2\rangle - 2yy'\langle yy'\rangle + y'^2\langle y^2\rangle)/\langle\eta^2\rangle + r(x,y) \ .$$

Finally, we have to investigate the precondition for the validity of this approach, i.e. that $r(x,y)$ has to be invariant versus the canonical transformation (7). As can be seen immediately, it is fulfilled for any $\chi(s)$, if either

- $K = 0$, i.e. the space charge forces are negligible, or
- $U(x,y) = cx^2 + dy^2$, i.e. the space charge potential is quadratic before and after the transformation (7). This is only true if the underlying phase space distribution is of the K-V type.

In both cases, the function $r(x,y)$ vanishes identically. The equations of motion are linear then, and (7) gives us the correlation between the harmonic oscillation of the beam particles inside a constant focusing system and the "pseudo-harmonic" particle oscillations occurring during the beam propagation through an arbitrary periodic focusing system.
In all other cases, the precondition can only be fulfilled if $\chi(s) = 0$, i.e. if the phase advances in both systems agree. Otherwise, the space charge potential

$U(x,y;s)$ of the distribution $f(x,y,x',y';s)$ would be a function of the transverse angles ξ' and η' of the particles forming the distribution $\phi(\xi,\eta,\xi',\eta')$. This means that the transformation (7) does not provide us any more with a time-dependent relationship of the individual particle's motion in both types of transport systems. The existence of this restriction is not surprising, since this relationship, which is not linear for non-K-V distributions under space charge conditions, cannot be supplied by a linear transformation as given by (7).
On the other hand, it is sufficient to show that the canonical transformation (7) establishes the correlation between a constant and the appropriate periodic focusing channel at fixed positions s along the transport lines. The distribution function f is a stationary solution of Vlasov's equation (6) pertaining to a periodic system, if f reproduces itself exactly passing through one focusing period. Since a stationary distribution ϕ in the constant focusing system is a constant of motion, a necessary and sufficient condition for the distribution function f to be S-periodic and hence stationary is that the beam transport through both systems can be correlated at integer multiples of one focusing period. If the phase advances of both systems agree at these points:

$$\chi(s) = 0 \iff s = s_0 + nS \quad , \quad n = 0,1,2,\ldots \quad ,$$

the transformation (7) represents this correlation. Therefore, for non-K-V distributions under space charge conditions, the two types of beam transport systems must be "equivalent", i.e. the phase advances over the distance S must agree for distributions correlated by (7) and carrying the same current:

$$\sigma := \psi_p(S) = \psi_c(S) \quad .$$

Due to the correspondence of the beam emittances associated with both distributions, f and ϕ, under these circumstances the zero current phase advances agree as well:

$$\sigma_0 := \psi_p^0(S) = \psi_c^0(S) \quad .$$

If we transform Poisson's equation

$$\Delta V(\xi,\eta) = -4\pi \cdot \iint \phi(\xi,\eta,\xi',\eta')\, d\xi' d\eta'$$

at positions, where $\chi(s) = 0$ via (7) into the new coordinate system, it follows that

$$V(\xi,\eta) = U(x,y;s) \quad ,$$

if the beam keeps its azimuthal symmetry. Consequently, the precondition is fulfilled for round beams as they occur in interrupted solenoid channels, where any point s_0 within the focusing period can be chosen to match the beam from the constant to the equivalent periodic system. Both types of transport systems are thus correlated by (7) point-by-point along the the focusing period, and C represents indeed the constant of motion for the periodic system.
In quadrupole channels of the FODO-type, there are only two points within one focusing period, where the envelopes agree in both transverse directions, i.e, where the beam is azimuthally symmetric. Therefore, matching has to take place at these specific axial positions s_0, in order to conserve Poisson's equation strictly. Fig. 1 shows the calculated emittance growth factors versus the number of periods of the GSI quadrupole channel[5] for three different initial phase space distribution functions. For a comparison, the growth factors as they would occur for an equivalent constant focusing channel are included in this figure.
The "geometrical" waterbag distribution (i.e. a homogeneous filling of the phase space within an ellipsoidal boundary), which is not self-consistent under space charge conditions, shows a rapid emittance growth due to a reduction of the non-linear space charge field energy[8] in both types of transport systems.

The self-consistent waterbag distribution as well as the K-V type shows practically no emittance growth in both cases, periodic quadrupole channel (p) and equivalent constant focusing channel (c). Since the results even for the K-V distribution deviate from the theoretical value of 1.0, the remaining emittance growth, which in this case amounts to about 0.2% after 15 periods, can be attributed to the limited accuracy of the computer simulations.

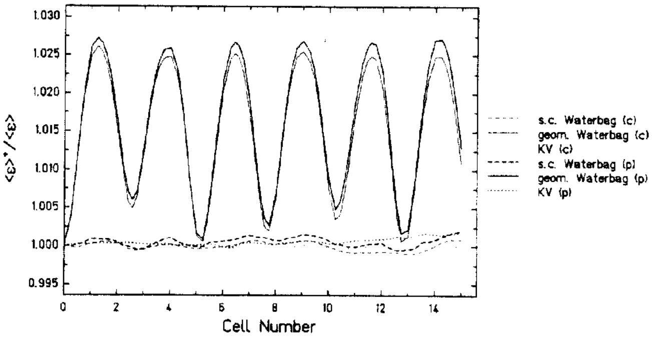

Fig.1. Emittance growth factors versus the number of periods obtained from particle simulations for initial K-V, self-consistent and "geometrical" waterbag distributions for the GSI quadrupole channel (p) and the equivalent constant focusing channel (c) at $\sigma_0 = 60°$, $\sigma = 25°$

4. Conclusions

The theory described here extends the work done by Sacherer[7], who first derived the envelope equations for continuous beams which were not restricted to the K-V distribution. Yet for not self-consistent particle phase space distributions they are of little practical use, since "the time dependence of the rms emittance must be known a priori" (Sacherer). On the other hand, we are interested to know the conditions where the rms emittance becomes a constant of motion. Extending the concept of equivalent beams (i.e. beams whose second moments and currents agree), the idea of "equivalent beam transport systems" has been introduced. Beam transport systems carrying matched beams with corresponding rms emittances and beam currents are called "equivalent", if they yield the same generalized phase advances along the focusing period S as well as along the distance S for the constant focusing system. For a constant focusing channel it is possible to construct arbitrary self-consistent non-K-V distributions. If these distributions are rms-matched to an equivalent periodic focusing channel, they show the same self-consistent behavior. This matching transformation conserves Poisson's as well as Vlasov's equation strictly, if the beam keeps its azimuthal symmetry. Under these circumstances, the rms envelope equations form a closed set to describe the evolution of the rms size of the beam, owing to the fact that the emittance is a constant of motion. As has been verified by computer simulations, we obtain non-K-V distributions this way, that conserve their rms emittance under space charge conditions even if passing through a periodic quadrupole channel.

References

[1] Lapostolle, P.M., CERN, ISR/300 LIN/60.19. (1969)
[2] Gluckstern, R.L., Chasman, R., Crandall, K., Linac Conference, FNAL, p. 823 (1970)
[3] Hofmann, I., INS-NUMA-45, p. 4 (1982)
[4] Reiser, M., Part. Acc. 8, p. 167 (1978)
[5] Struckmeier, J., Thesis, to be published
[6] Leach, P.G.L., SIAM J. Appl. Math. 34, No.3, p. 496 (1978)
[7] Sacherer, F.J., IEEE Trans. Nucl. Sci. NS-18, p. 1105 (1971)
[8] Struckmeier, J., Klabunde, J., Reiser, M., Particle Accelerators 15, p. 47 (1984)