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SELF-CONSISTENT TREATMENT OF EQUILIBRIUM SPACE CHARGE EFFECTS IN THE ℓ=2 STELLATRON

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A cyclic accelerator, the stellatron[1,2,3], has been proposed to accelerate large electron currents in a magnetic field focusing configuration consisting of the combination of an ordinary weak focusing betatron field, a toroidal field, and a stellarator field. Early treatments[1] of space charge effects in the stellatron employed the ad hoc assumption of a circular beam of arbitrary, unspecified radius. In the present work we obtain an equilibrium distribution function for the ℓ=2 stellatron, analogous to the K-V distribution function, which produces an elliptical beam of constant density, the major and minor radii and orientation of which are specified by the beam current and emittance, and by the externally applied fields. Single particle stability criteria and single particle tunes in the self-consistent fields are also given.

Two basic types of stellatron have been proposed, the difference being in the ℓ-number, where ℓ is the number of field periods in the poloidal direction of the stellarator winding. Only the ℓ=0 and ℓ=2 devices, however, possess a finite magnetic field gradient on the central axis and so only these give a first order focusing field.

In the ℓ=o device[2] consideration of equilibrium space charge effects is most simply done in the Larmor frame in which the two transverse degrees of freedom are decoupled; in this frame the usual K-V envelope equations are applicable and these give the parametric dependencies of the beam size on current, beam emittance, and applied fields.

The situation is not as simple in the ℓ=2 stellatron[1], which is illustrated in Figure 1.

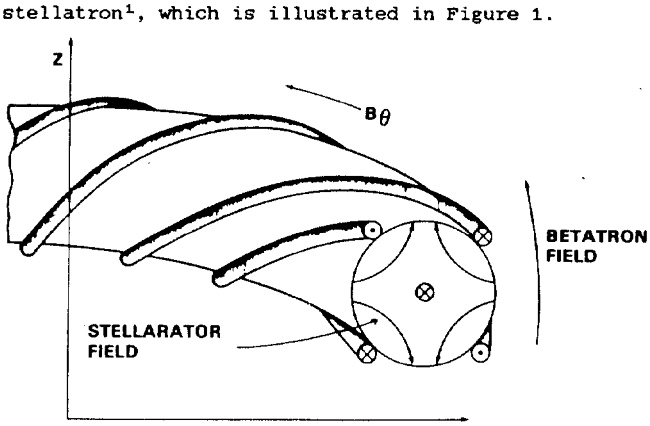

Fig. 1: Cutaway schematic view of the ℓ=2 stellatron illustrating the magnetic fields.

The continuous, twisted quadrupole focusing in the ℓ=2 device has the effect of coupling both transverse degrees of freedom. Here we shall show that it is possible nonetheless to carry out an analysis in a self-consistent way by explicitly constructing a 4-D analog of the K-V distribution function. In addition to the explicit relations it yields, the distribution function may also be useful as initial data in analytical or numerical investigations of stability questions.

An analysis similar to the one presented here has been carried out by Gluckstern[4] for a linear beam transport line.

Particle Motion in the ℓ=2 Stellatron

We consider an electron of energy $m_e\gamma_o c^2$ moving in a circular orbit of radius r_o in the $\hat{\theta}$ direction in the z=0 plane in a vertical magnetic field B_{zo}. Near this orbit we take the applied magnetic fields in a standard (r,θ,z) cylindrical coordinate system to be

$$\begin{aligned} B_r &\simeq B_{zo}\,[-ny + \mu(y\cos m\theta - x\sin m\theta)] \\ B_\theta &\simeq B_{\theta o} \\ B_z &\simeq B_{zo}\,[1-nx + \mu(x\cos m\theta + y\sin m\theta)] \end{aligned} \tag{1}$$

where B_{zo} and $B_{\theta o}$ are constants, n is the usual betatron field index, x and y are the dimensionless displacements from the circle $r=r_o$: $x = (r-r_o)/r_o$, $y = z/r_o$, μ is a dimensionless measure of the quadrupole strength, and m is an integer, the number of quadrupole field periods around the circle.

We assume that the beam itself is of elliptical cross section and constant density centered on the point $r=r_o$, z = 0 and having axes of length a and b. The axis of length a is assumed to be tilted at an angle α from the $\hat{r}$ (or $\hat{x}$) direction.

Using the applied fields of (1) and the self fields of a tilted elliptical beam, the paraxial equations of motion of a particle in the beam are

$$\begin{aligned} & x'' + \left[1-n + \mu\cos m\theta - 2n_s\,\frac{b\cos^2\alpha + a\sin^2\alpha}{a+b}\right] x \\ & \quad = -\left[\mu\sin m\theta + 2n_s\,\frac{a-b}{a+b}\cos\alpha\,\sin\alpha\right] y + b_o y' \\ & y'' + \left[n - \mu\cos m\theta - 2n_s\,\frac{a\cos^2\alpha + b\sin^2\alpha}{a+b}\right] y \\ & \quad = -\left[\mu\sin m\theta + 2n_s\,\frac{a-b}{a+b}\cos\alpha\sin\alpha\right] x - b_o x' \end{aligned} \tag{2}$$

where $b_o = B_{\theta o}/B_{zo}$, $n_s = \omega_b^2/(2\gamma_o^2\Omega_{zo}^2)$, ω_b is the beam plasma frequency, $\omega_b^2 = -4e\lambda/(m_e\gamma_o ab)$, -e is the electron charge, $\Omega_{zo} = eB_{zo}/(m_e\gamma_o c) = \beta_o c/r_o$ is the basic electron cyclotron frequency and a prime denotes $d/d\theta$. At this point we have made no assumptions as to the θ-dependence of a,b, and α and in general these are unknown.

To make progress we shall consider the special case $n=\frac{1}{2}$. We expect the results to be insensitive to the exact choice of n since the focusing will be dominated by the rotating quadrupole term (μ will be much larger than n in general). Choosing $n=\frac{1}{2}$, then, and defining $\xi=x+iy$, the two equations in (2) may be expressed as

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$$\xi'' + ib_o\xi' + \left[\frac{1}{2} - n_s\right]\xi + \left[\mu e^{im\theta} + n_s\,\frac{a-b}{a+b}\,e^{2i\alpha}\right]\xi^* = 0 \tag{3}$$

The form of (3) suggests that we consider the following behavior of the beam radii and orientation

$$\begin{aligned} a &= \text{constant} \\ b &= \text{constant} \\ \alpha &= m\theta/2 \end{aligned} \tag{4}$$

which corresponds to a beam of fixed size aligned and twisting with the axes of the quadrupole. A beam satisfying (4) will be called a matched beam.

For a matched beam Eq. (3) may be solved by defining a new variable in a coordinate system rotating with the quadrupole and beam axes. If we let $\xi = \psi\exp(im\theta/2)$ we find

$$\psi = A_+e^{i\nu_+\theta} + \sigma_+A_+^*e^{-i\nu_+\theta} + A_-e^{i\nu_-\theta} + \sigma_-A_-^*e^{-i\nu_-\theta} \tag{5}$$

where $A_\pm$ are arbitrary complex constants, $\nu_\pm$ are the positive roots of

$$\nu_\pm^2 = \hat{n} + \frac{1}{4}\hat{m}^2 \pm (\hat{n}\hat{m}^2 + \hat{\mu}^2)^{\frac{1}{2}} \tag{6}$$

where $\hat{m}=m+b_o$, $\hat{n}=\frac{1}{2}-n_s+b_o^2/4$, $\hat{\mu}=\mu+n_s(a-b)/(a+b)$,

and

$$\sigma_\pm = \frac{\left[\nu_\pm + \frac{1}{2}\hat{m}\right]^2 - \hat{n}}{\hat{\mu}} = \frac{\hat{\mu}}{\left[\nu_\pm - \frac{1}{2}\hat{m}\right]^2 - \hat{n}}\,. \tag{7}$$

When they are real, $\nu_\pm$ are clearly the number of oscillations a particle executes when traveling around the stellatron. We will sometimes refer to the + and - oscillation modes as the fast and slow modes, respectively. Both are stable ($\nu_\pm^2$ is real and positive) if and only if the following three conditions are simultaneously satisfied:

$$\begin{aligned} &\hat{n}\hat{m}^2 + \hat{\mu}^2 \geq 0 \\ &\hat{n} + \frac{1}{4}\hat{m}^2 \geq 0 \\ &|\hat{n} - \frac{1}{4}\hat{m}^2| \geq |\hat{\mu}|\,. \end{aligned} \tag{8}$$

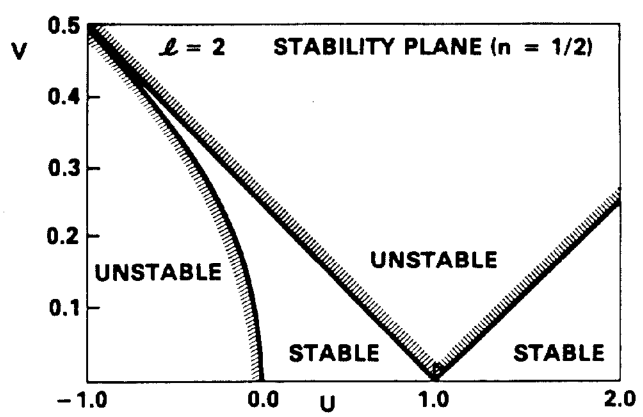

Fig. 2: Single particle stability plane including self field effects. u and v are defined in the text.

In the absence of space charge effects only the last of these is nontrivial. When space charge effects are present we may illustrate (8) in a plane of two auxiliary variables, $u=4\hat{n}/\hat{m}^2$, $v=|\hat{\mu}|/\hat{m}^2$ as shown in Figure 2. Any experiment must be designed to operate wholly within one or the other of the stable regions shown in the figure. In the case a=b all results to this point, including Figure 2, reduce to those obtained in reference [1].

We shall consider a distribution function[4] dependent only on the fast and slow mode amplitudes $|A_\pm|$, which we may express in terms of ψ and ψ' using (5). Denoting by subscripts r and i the real and imaginary parts of ψ we find

$$\begin{aligned} |A_+|^2 &= \frac{1}{D_1^2}\left[\nu_-(1-\sigma_-)\psi_r - (1+\sigma_-)\psi_i'\right]^2 \\ &\quad + \frac{1}{D_2^2}\left[\nu_-(1+\sigma_-)\psi_i + (1-\sigma_-)\psi_r'\right]^2 \\ |A_-|^2 &= \frac{1}{D_1^2}\left[\nu_+(1-\sigma_+)\psi_r - (1+\sigma_+)\psi_i'\right]^2 \\ &\quad + \frac{1}{D_2^2}\left[\nu_+(1+\sigma_+)\psi_i + (1-\sigma_+)\psi_r'\right]^2 \end{aligned} \tag{9}$$

where

$$\begin{aligned} D_1 &= \nu_-(1+\sigma_+)(1-\sigma_-) - \nu_+(1-\sigma_+)(1+\sigma_-) \\ D_2 &= -\nu_-(1-\sigma_+)(1+\sigma_-) + \nu_+(1+\sigma_+)(1-\sigma_-). \end{aligned} \tag{10}$$

The distribution function we choose is analogous to the K-V distribution,

$$f(\psi_r,\psi_r',\psi_i,\psi_i') = f_o\delta[f_+|A_+|^2 + f_-|A_-|^2 - 1] \tag{11}$$

where f_o, $f_\pm$ are constants, independent of ψ and ψ' and δ is the Dirac delta function. The distribution function (11) has the well known feature that its integral over any two of the four variables $\psi_r,\psi_r',\psi_i,\psi_i'$, considered as a function of the remaining two, vanishes outside of an ellipse and is constant inside the ellipse. For example, integration of (11) over ψ_r' and ψ_i', using (9), gives expressions for the beam radii:

$$\begin{aligned} a^2 &= \frac{r_o^2}{f_+f_-}\left[f_+(1+\sigma_-)^2 + f_-(1+\sigma_+)^2\right] \\ b^2 &= \frac{r_o^2}{f_+f_-}\left[f_+(1-\sigma_-)^2 + f_-(1-\sigma_+)^2\right] \end{aligned} \tag{12}$$

A circular beam results when $f_+\sigma_- + f_-\sigma_+ = 0$, which corresponds to the treatment in reference [1]. The expressions in (12) are not generally useful, however, until we specify the unknowns $f_\pm$. These constants may be related to the (unnormalized) beam emittances by integrating (11) over ψ_i and ψ_i' and then over ψ_r and ψ_r'. One finds

$$\epsilon_r^2 = \frac{a^2}{f_+ f_-}\left[f_+\nu_-^2\,(1+\sigma_-)^2 + f_-\nu_+^2\,(1+\sigma_+)^2\right]$$

$$\epsilon_i^2 = \frac{b^2}{f_+ f_-}\left[f_+\nu_-^2(1-\sigma_-)^2 + f_-\nu_+^2(1-\sigma_+)^2\right] \qquad (13)$$

where $\pi\epsilon_r$ is the area in the $r_o\psi_r$, ψ_r' plane and $\pi\epsilon_i$ is the area in the $r_o\psi_i$, ψ_i' plane. It is possible to specify the emittances ϵ_r and ϵ_i, solve (13) for f_+ and f_-, and substitute in (12) to obtain (rather complicated) implicit expressions for a and b. (Recall that ν and σ depend on a and b.) In one particular case, though, this is easily carried out.

The Case $\hat{m} = 0$

When $\hat{m} = m + b_o = 0$ the quadrupole axes are rotated at a spatial rate equal to the electron Larmor frequency in the toroidal field divided by the electron velocity, $\alpha' = -r_o\Omega_{\Theta o}/2V_{\Theta o}$, where $\Omega_{\Theta o} = eB_{\Theta o}/m_e\gamma_o c$. The sense of the quadrupole rotation about the toroidal field lines in this case is right-handed, that is, in the same sense as ordinary electron gyromotion about the toroidal field lines.

When $\hat{m}=0$ the two degrees of freedom, ψ_r and ψ_i in the rotating frame are decoupled. One easily finds $\sigma_\pm = \pm 1$ and the relations governing the beam radii,

$$(\epsilon_r r_o/a^2)^2 = \frac{1}{2} + \frac{1}{4}b_o^2 + \mu - \frac{2b}{a+b}n_s \qquad (14)$$

$$(\epsilon_i r_o/b^2)^2 = \frac{1}{2} + \frac{1}{4}b_o^2 - \mu - \frac{2a}{a+b}n_s$$

which of course are just the relations which would be obtained from the usual K-V envelope equation if written in the Larmor frame of reference.

In Figure 3 we have plotted beam radii and fast and slow mode tunes versus current using (14) for the case $b_o = \mu = 0$ (i.e. a conventional betatron). Since the focusing is symmetric, a=b. The zero current value of the tune is just $\sqrt{n} = \sqrt{\tfrac{1}{2}}$. (In these and subsequent figures we have taken $\epsilon_r = \epsilon_i = 0.5$ rad-cm, $r_o = 100$ cm, and $mc^2(\gamma_o - 1) = 1$MeV.)

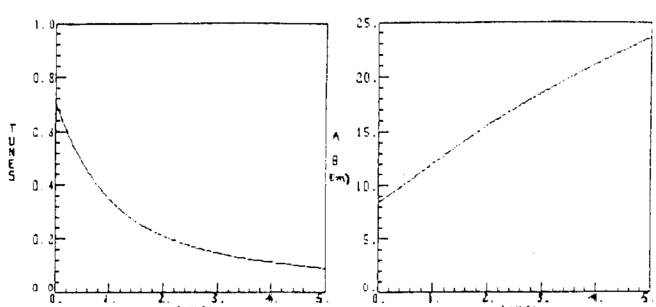

Fig. 3: Beam radius and tune versus current for a conventional betatron.

In Figure 4 we have plotted the same quantities (in the Larmor frame) for a betatron with an added toroidal field ($B_{\Theta o}$ = 1 kG, μ=0) and in Figure 5 we have shown, again using (14), the case ($m=-b_o=20$, μ=50). We see that although the beam radii are naturally much larger in the weak focusing conventional betatron, the tune shifts due to space charge are potentially much more serious in the modified betatron and stellatron devices. Currents may be limited to modest values in these, as in other strong focusing devices, by the tune shift effect unless dangerous resonant (especially, integer) values of $\nu_\pm$ are somehow avoided or passed through quickly enough.

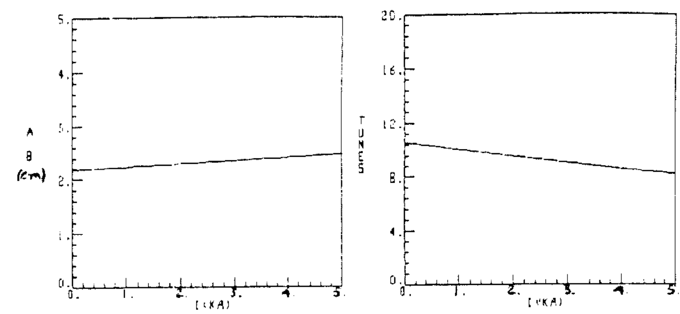

Fig. 4 Beam radius and tune in Larmor frame versus current for a modified betatron.

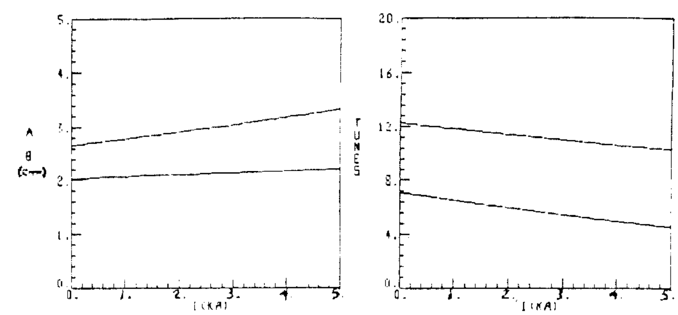

Fig. 5: Beam radii and tunes versus current for an ℓ=2 stellatron ($\hat{m}$=0 , μ=50).

Acknowledgments

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