IEEE Transactions on Nuclear Science, Vol. NS-32, No. 5, October 1985

NUMERICAL SOLUTION OF BOUNDARY CONDITION TO POISSON'S EQUATION AND ITS INCORPORATION INTO THE PROGRAM POISSON* †

S. Caspi, M. Helm, and L.J. Laslett
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

Summary

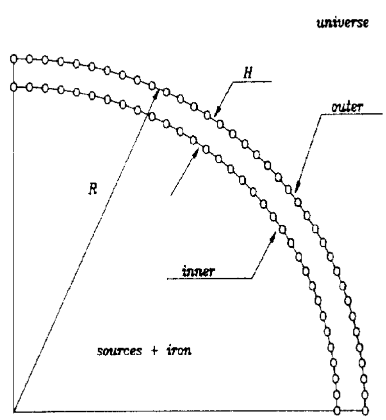
Two dimensional cartesian and axially-symmetric problems in electrostatics or magnetostatics frequently are solved numerically by means of relaxation techniques -- employing, for example, the program POISSON. In many such problems the "sources" (charges or currents, and regions of permeable material) lie exclusively within a finite closed boundary curve and the relaxation process in principle then could be confined to the region interior to such a boundary -- provided a suitable boundary condition is imposed onto the solution at the boundary. This paper discusses and illustrates the use of a boundary condition of such a nature in order thereby to avoid the inaccuracies and more extensive meshes present when alternatively a simple Dirichlet or Neumann boundary condition is specified on a somewhat more remote outer boundary.

Introduction

The proposed boundary condition may be most simply illustrated by specific use of plane-polar coordinates. Thus, with a circular boundary so located that no external sources are present, the potential function external to that boundary is expressible in the form

$$C_0 + \sum_{m=1}^{\infty} r^{-m} \left(C_m \cos m\theta + S_m \sin m\theta \right),$$

in which no positive powers of r occur. Such a relation will permit one to extend the potential to a surrounding concentric circle of somewhat larger radius. If, in practice, values of potential are known at only a finite number of points on the inner circle, then of course only a finite number of harmonic coefficients (C_m,S_m) could be evaluated for such trigonometric representation of the potential function -- such a trigonometric series may, however, be adopted to provide adequate estimates of the corresponding values of potential at various points on a near-by surrounding "outer-boundary curve".



XBL 847-3026

In performing a relaxation computation on a mesh bounded by such a pair of curves (external to all "sources"), any full relaxation pass through the mesh may be followed by a step wherein the values of potential at points on the outer boundary are revised (up-dated) on the basis of a harmonic description of the potential function on the inner curve. Such revised values would then be employed, as boundary values, in proceeding with the next relaxation pass through the mesh. [An analogous procedure of course would be followed if one were to adopt an elliptical coordinate system (u,v), for which harmonic terms would be of the form e^{-mu} times circular functions of argument mv].

In the work summarized here, we have made a practical application of the techniques just described, with particular application to the use of the relaxation program POISSON as applied to the design of superconducting magnets for advanced particle accelerators. It is evident that in such work one takes advantage of such intrinsic symmetries as may be present in the geometrical configuration and current distribution for the problem of interest. One realizes also that, in practice, there may be a large number of mesh points along the inner (circular) curve whereon one constructs a harmonic representation of the potential and (especially for circular boundaries) such points may have a quite unequal spacing. Under such circumstances it may well be expedient, as we indicate, to base the analysis on a restricted number of trigonometric coefficients and to compute these coefficients by a weighted least-squares evaluation of the data.

The following note includes a description of the equations introduced into our operating POISSON program, and this material is followed by some illustrative examples.

Analysis

Consider the case where a circular arc of radius $r = R - H$ divides space into two regions, an inner one which includes all current sources and magnetic iron, and an outer one which is in free space (hereafter referred to as the "universe"). Since the free space region is infinite we shall arbitrarily limit it by a secondary circular arc of radius $r = R$. Both circular arcs are an assembly of connecting mesh points such as the one generated by the program LATTICE. If we know the vector potential for each mesh point on $r = R - H$ (e.g. calculated by POISSON), we would like to find the vector potential at each mesh point on $r = R$, so that such values may be employed as provisional boundary values in a subsequent relaxation pass through the entire mesh. This is expressed as:

$$A_k^{outer} = \sum_{n=1}^{N} E_{kn} A_n^{inner} \tag{1}$$

A is the vector potential and E is a working matrix, and the summation is over the entire mesh points of the inner arc.

*This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, High Energy Physics Division, U.S. Dept. of Energy, under Contract No. DE-AC03-76SF00098.

†Development of the work presented here has been summarized in LBL Reports LBL-17064 (formerly LBID-887), LBL-18063, LBL-18798, and LBL-19050 (Lawrence Berkeley Laboratory, 1984-85).

0018-9499/85/1000-3722$01.00© 1985 IEEE

In the free space region the vector potential can be expressed as a sum of harmonic terms, each employing powers of 1/r.

$$A_i = \sum_{\ell=1}^{\infty} r^{-\alpha_\ell} D_\ell F_\ell(\theta_i) \qquad (2)$$

The vector potential A of mesh point i on the circular arc r is expressed in terms of a series of functions F_ℓ (θ), their coefficients D_ℓ and the problem type symmetry α_ℓ.

Summing over the N boundary points on the radius r, the difference between the calculated vector potential values and the relaxed ones is minimized with respect to D_ℓ.

$$\text{Min:} \quad 1/2 \sum_{i=1}^{N} W_i \left(\sum_{\ell=1}^{m} r^{-\alpha_\ell} D_\ell F_\ell(\theta_i) - A_i \right)^2 \qquad (3)$$

The number of harmonic terms has been reduced to m and the weight factors W_i have been introduced to take care of an uneven distribution of mesh points along the boundary.

Following the minimization process we arrive at:

$$\sum_{j=1}^{m} M_{ij} D_j r^{-\alpha_j} = V_i \qquad (4)$$

where:

$$M_{ij} = \sum_{n=1}^{N} W_n F_i(\theta_n) F_j(\theta_n)$$

$$i, j = 1, 2, 3 \ldots m$$

$$V_i = \sum_{n=1}^{N} W_n F_i(\theta_n) A_n$$

Solving for D_j on the inner arc r = R - H we get

$$D_j = \sum_{i=1}^{m} (R-H)^{\alpha_j} (M^{-1})_{ji} V_i^{inner} \qquad (5)$$

Using Eq. (2) on the outer arc r = R and substituting the expressions for D_j and V_i we arrive at (Eq. 1)

$$A_k^{outer} = \sum_{n=1}^{N} E_{kn} A_n^{inner}$$

where

$$E_{kn} = \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\frac{R-H}{R} \right)^{\alpha_j} W_n (M^{-1})_{ji} F_j(\theta_k) F_i(\theta_n)$$

We put an arbitrary upper limit on the number of harmonics m $\leq$ 50.

Two Dimensional Case with Plane-Polar Coordinates

The harmonic functions F_ℓ (θ) are a combination of the trigonometric functions SIN and COS. It is, however, convenient to express them in the following way

$$F_\ell(\theta) = \cos\left(\alpha_\ell \theta - \beta_\ell \frac{\pi}{2}\right)$$

α and β are integer division of : $\alpha = \frac{\ell}{2}$; $\beta = \frac{\ell}{2} - \frac{\ell-1}{2}$

Two Dimensional Problems with Elliptical Cylindrical Coordinates

We replace the two circular arcs with two confocal ellipses and employ elliptic cylindrical coordinates.

$$\left(\frac{R-H}{R} \right)^{\alpha_j} = \left[\frac{(a+b)_1}{(a+b)_2} \right]^{\alpha_j}$$

$$F_\ell(v) = \cos\left(\alpha_\ell v - \beta_\ell \frac{\pi}{2}\right)$$

a and b are the semi-axes and $v = \tan^{-1}[(y/x)/(b/a)]$.

Axis-Symmetry Problems with Polar Coordinates

Here we consider cases which posses symmetry with respect to revolution around the Z axis. In a cylindrical geometry the flux lines are represented by the product ρA_ϕ, where $\rho = r\sin\theta$. The program POISSON is written in such a way that this product is the one which is being relaxed.

$$F_\ell(\Theta) = \frac{\sin\theta \, P_{\alpha_\ell}^1(\cos\theta)}{\alpha_\ell} \quad ; \quad \alpha_\ell = \ell, \quad -1 \leq \cos\theta \leq 1$$

P_n^1 (u) are the associated Legendre functions.

Axisymmetrical Problems with Prolate Spheroidal Coordinates

We replace the circular arcs with two confocal ellipsoids. It then becomes permissible to introduce terms in a development of A_ϕ that involve

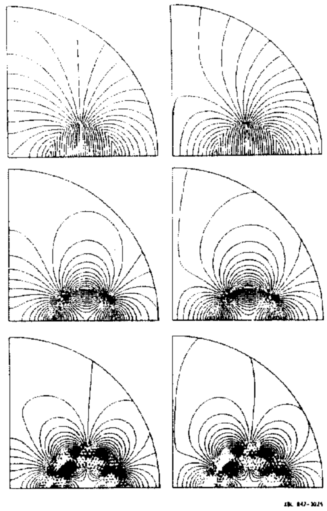
$$F_\ell(v) = \frac{\sin v \, P_{\alpha_\ell}^1(\cos v)}{\alpha_\ell}$$

$$\left(\frac{R-H}{R} \right)^{\alpha_j} = \left(\frac{a_{inner}}{a_{outer}} \right)^{\alpha_j} \frac{H_{\alpha j}(\eta_{outer})}{H_{\alpha j}(\eta_{inner})} \quad ; \quad \eta = \frac{a}{c}$$

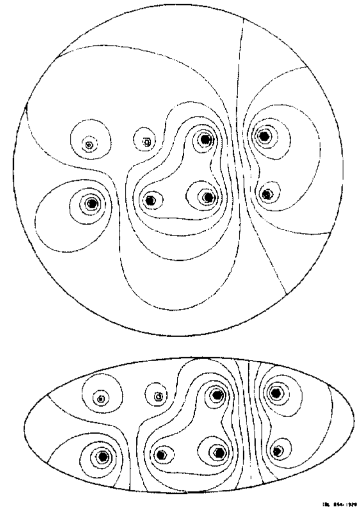
$H\alpha_j(\eta)$ is a normalized function derived from the associated Legendre function Q_n^1 (η), η is the eccentricity, and $c = (a^2 - b^2)^{1/2}$.

Superposition

If an externally imposed field or a known external circulation is present, the vector potential representing such contributions can be subtracted from the potential on the inner circular curve prior to applying the boundary relations presented here. Corresponding potential values for the external field or circulation then are added to the values so transferred to the outer curve.

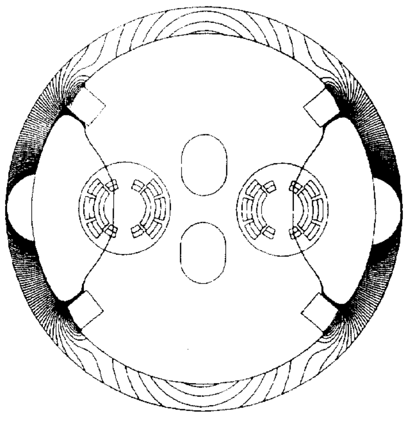


(a)

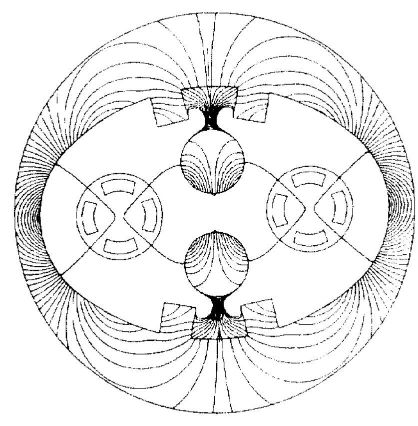


(b)

POISSON relaxation of two dimensional cartesian problem of various symmetries (a), and no symmetry (b), using both circular and elliptical boundaries. Case (b) was checked and found to be in good agreement with analytical calculation.

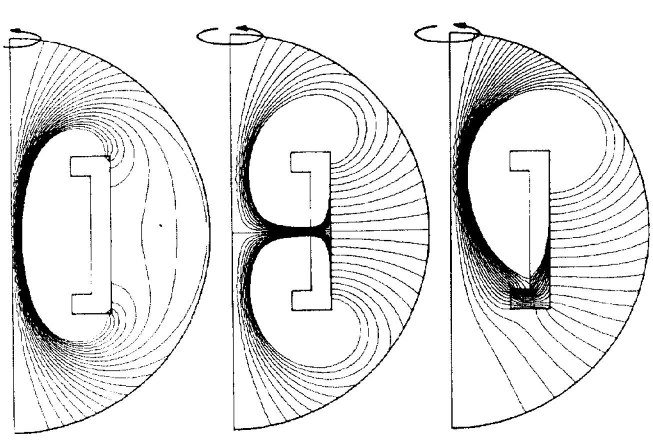


(c)

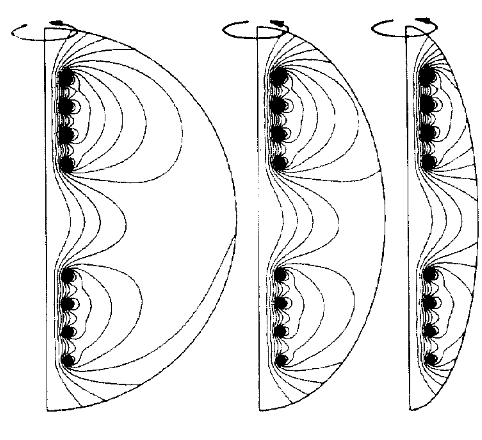


(d)

Relaxed flux lines -- in a SSC dipole (c) and quadrupole (d) -- are magnified along the boundary by choosing only lines which leak out from the iron (Reference Design A).



(e)



(f)

POISSON relaxation of axisymmetrical problems -- including iron and possible symmetries. Selected flux along the boundary is plotted in case (e). Case (f), for both circular and elliptical boundaries, was checked and found to be in good agreement with analytical calculation.