The Higher Order Modes Calculation of RF Cavity With Finite Element Method

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Abstract

A previous paper has presented to calculate not only the fundamental and longitudical modes but also the transverse higher modes of RF cavity. But with this method higher modes are calculated one by one.

In this paper we present a finite element method using triangulated elements, six points quadratic interpolation for improvement. At first, we obtain a tridiagonal coefficient matrix with the Householder method, then calculate its eigenvalues using half-divided method. With this method we can calculate any order eigenvalues(and resonant frequency values) and the responding eigenvectors (and field distribution) of a mode group simultaneously at one time. This method is convenient, accurate and economical for computing time.

Introduction

To calculate the higher order modes with finite differentiation succesive over relaxation method, the higher order eigenvalues are solved with high accuracy, but the computing time is not economical because the higher modes are calculated one by one. Superfish method is popular, but it can evaluate longitudinal modes only and takes lots of computing time also.

This paper represents a finite element method to calculate the resonant frequencies and the field distribution of a symmetric RF cavity. The Householder method is used to form a tridiagonal matrix and then evaluate the eigenvalues with half-divided method. The essential advantage of this method is that any eigenvalues can be obtained simultaneously. It makes calculation convenient and economical for computing time. But, if we simply make linear interpolation on each triangular element, the accuracy is lower. In this paper the quadratic interpolation method is proposed. With this method the accuracy is improved obviously and the computing time is shortened. Practical computation proves that this method is worth being adopted.

The Solution of The Generalized Algebra Eigenvalue Problem

In this paper we use the finite element method in which the basic principle, fundamental equations and notations are the same as the previous paper (1).

To derive from the Hertz vector wave equation and the boundary condition, the equivalent variation problem can be obtained:

$$F(U)=\int_S\left[\frac{n^2}{r^2}U^2+\left(\frac{\partial U}{\partial r}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2\right]rdrdz - \int_S K^2U^2rdrdz = \min.$$

$U|_{r=0}=0$ (for $n\neq 0$)

$U|_\Gamma=0$ (for Γ not perpendicular to Z axis (TM mode), for Γ not parallel to Z axis (TE mode))

where Γ represents the surface of perfect conductor. Then the basic finite element equation can be obtained:

$$AU=\lambda BU$$

A and B both are n order, symmetric and positive definite coefficient matrix. U is a column matrix consisted of the potential functions of n nodes. λ is the eigenvalue of the matrix, and relates with K: $\lambda=K^2$.

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Because matrix B is symmetric and positive definite, it can be resolved by a product of a lower triangle matrix L and its transposed matrix L^T by the square root method.

$$B = LL^T$$

Then $L^{-1}AL^{-T}L^TU = \lambda L^TU$

where $L^{-T} = (L^{-1})^T$.

Let $P = L^{-1}AL^{-T}$ and $X = L^TU$.

Then $PX = \lambda X$.

By means of this transformation, the problem of finding the generalized eigenvalue is then transferred to finding the general eigenvalue.

Since the matrix P is analogous to the matrix A, the eigenvalues of P are the same as the eigenvalues of A.

After orthogonal analogous transformation $(F^T = F^{-1})$ by n-2 times with the Householder method. Matrix P can be transferred to a symmetric tridiagonal matrix:

$$Q = FPF^T.$$

And then the equation can be changed to:

$$FPF^{-1}FX = \lambda FX .$$

Let $Y = FX$,

then $QY = \lambda Y$.

The half-divided method can be used to solve the eigenvalues of this tridiagonal matrix. After an approximate eigenvalue λ_i' has been obtained, the accurate eigenvalue λ_i and the corresponding eigenvector can be obtained with the inverse power method. Under the circumstances of the λ_i' value being known, the convergence rate of this method is high, and any eigenvalues can be calculated according to the requirement.

Using the eigenvalue got above, the corresponding resonant frequency:

$$f = c\sqrt{\lambda}/2\pi .$$

The field distribution can be obtained after transformating the eigenvector inversely for several times.

Improving Accuracy With Triangular Element Six Points Quadratic Interpolation

The linear interpolation on a triangular element is simple and convenient but the accuracy is not satisfied. For improving the accuracy the number of nodes may be increased but the computing time will be increased greatly. In this paper, six points quadratic interpolation on triangular element is adopted.

Taking six interpolation nodes on a triangular element, three apexes and three middle points, the following multinomial function can be used:

$$U(r,z) = a_1 + a_2 r + a_3 z + a_4 r^2 + a_5 rz + a_6 z^2$$

The surface integral can be obtained by numerical integration with area coordinates. With this improving method, the accuracy will be increased obviously even fewer elements.

The Results of Calculation

For checking the accuracy of calculation, we calculate an empty cylindrical cavity which can be calculated analytically in theory. For the sake of the economization of computing time, the field region is cut sparely, and the number of the nodes is not over 42. The calculating results with the quadratic interpolation method is still accordant with analytic solution fairly well.

In table 1, the resonant frequencies of variety of modes are presented, such as transverse modes and longitudinal modes, TE modes and TM modes, and modes of odd symmetry and of even symmetry. To compare the two methods, it is obvious that the quadratic interpolation method is more accurate.

With the quadratic interpolation method it takes no more than twenty minutes to calculate four modes at one time on a minicomputer Cromenco II.

From the calculating resuls, it proves that the quadratic interpolation method is effective and feasible.

Table 1. The Resonant Frequency of Cylindrical Cavity
(D = 76 mm , L = 68 mm)

Mode	Analytic solution(MHz)	Calculating value (MHz) Linear interpolation	Quadratic interpolation	Calculating error Linear interpolation	Quadratic interpolation
TM_{010}	3019.424	2850.104	3061.439	5.61×10^{-2}	1.39×10^{-2}
TM_{011}	3738.828	3506.399	3790.028	6.22×10^{-2}	1.57×10^{-2}
TM_{012}	5343.153	5057.562	5424.803	5.35×10^{-2}	1.51×10^{-2}
TM_{013}	7270.095	6948.157	7393.156	4.43×10^{-2}	1.67×10^{-2}
TE_{011}	5291.820	4861.295	5366.387	8.14×10^{-2}	1.41×10^{-2}
TE_{013}	8178.436	7594.823	8346.247	7.13×10^{-2}	2.05×10^{-2}
TE_{014}	9851.260	8663.227	9851.260	1.21×10^{-1}	2.68×10^{-2}
TM_{110}	4810.990		4813.887		6.02×10^{-4}
TM_{111}	5292.577		5298.638		1.15×10^{-3}
TM_{212}	7812.009		7851.206		5.02×10^{-3}
TM_{213}	9237.134		9328.819		9.93×10^{-3}
TE_{114}	9115.625		9215.642		1.09×10^{-2}
TE_{222}	9502.539		9617.918		1.21×10^{-2}
TE_{113}	7004.592		7030.134		3.73×10^{-3}

Table 2. The Field Distribution of TM_{010} Mode
(D = 76 mm, L = 68 mm)

Calculated with linear interpolation

r(mm)	0.633	1.267	1.900	2.533	3.167	3.800
H_ϕ	0.196	0.367	0.498	0.576	0.576	0.520
H_ϕ'	0.155	0.495	0.733	0.866	0.866	0.797
H_ϕ'/H_ϕ	0.800	1.340	1.470	1.500	1.540	1.530

Calculated with quadratic interpolation

r(mm)	0.700	1.400	2.100	2.800	3.300	3.800
H_ϕ	0.216	0.402	0.529	0.583	0.569	0.511
H_ϕ'	0.299	0.578	0.761	0.839	0.823	0.750
H_ϕ'/H_ϕ	1.384	1.438	1.439	1.439	1.447	1.468

$H_\phi : \propto J_1(\frac{U_{01}}{a}r)$, Analytic solution.
a = 3.8 mm.
H_ϕ': Calculating value.
r : Distance from the axis Z.

References

1. W.Zhou, X.W.Chu and M.D.Zhou, IEEE Trans. on NS, NS.30, No.4, August 1983, p.3627
2. K.Halbach and R.F.Holsinger, Partical Accelerators, 1976, Vol. 7, p. 213-222