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THE SPHERICAL RESONATOR

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One of the earliest microwave boundary value problems to be analyzed was electromagnetic oscillations within a spherical cavity.(1) This is curious since such resonators have found essentially no application in microwave engineering, doubtless partly for the reason that such resonators were supposedly difficult to fabricate.

It is the more curious since it followed less than twenty years H. Hertz' demonstration of the validity of Maxwell's wave equation and that electromagnetic waves propagated with the velocity of light. At that time, in contrast to the opinions of English physicists, continental physicists generally assumed that the far-field forces were transmitted instantaneously through space, the nature of which was of no importance in the transmission process.(2)

Similar other early analyses in spherical coordinates were largely mathematical exercises, in any case not intended for a projected application.(3) Treatments have also appeared in all standard texts.(4).

The conventional manner of solving the wave equation is separation, by which is meant that the partial differential equation of wave propagation is reduced to an ordinary differential equation in each coordinate. If the coordinate planes match the geometry of the volume the boundary conditions are then particularly simple to apply. Inconveniently, the wave equation is separable only in a few orthogonal, curvilinear coordinate systems, the so-called 'separable systems of Stäkel'.(5) There are, in fact, only eleven such Euclidean coordinate systems which allow separation of the scalar wave equation in three dimensions and five such systems for the vector wave equation.

It appears, therefore, that applicability of the separation technique is seriously limited and that there is need, consequently, to find other methods of solution. Such non-analytic methods have been developed in mesh-relaxation techniques.(6) On the other hand, even to this date separability has not been exhausted. A complete solution in this method is usually understood to include preparation of a table of values of the solution of the second order, linear differential equation arising from the separation technique. This process has been completed, of course, for the coordinate systems principally used.

In addition to the general method of separability and computer techniques there are some other artifices to avoid laborious or intractable equations. For example, a real physical solution of the wave equation must also satisfy Maxwell's equations; therefore some special solutions, usually for the lower order modes, may be found directly from the circulation equations.(7) Also, by analogy, on the basis of perturbation arguments, it is likely that certain oscillatory modes will exist in a cavity. For example, the existence of the TM-010 mode in a right circular cylinder (of height equal to the diameter) implies the existence of a similar mode in a spherical cavity of the same diameter. In fact, the TM-101 mode in a spherical cavity ($\lambda = 2.29\,a$, $Q = \eta/R_S$) resembles the TM-010 mode in a cylindrical cavity ($\lambda = 2.61a$, $Q = .8\eta/R_S$), a being the radius, η the impedance of free space and R_S the surface resistivity per square.(8)

The homogeneous wave equation,

$$\nabla^2 E = \frac{1}{\mu\epsilon}\,\frac{\partial^2 E}{\partial t^2} \tag{1}$$

for the total vector field is usually also true for one or more of the field components, depending on the coordinate system. For example, it is separately true for all components in rectangular coordinates, for the axial component only in cylindrical coordinates but not for any component in spherical coordinates. When this simplification can be made and separation is possible, solutions will be obtained in orthogonal functions; the remaining field components can then be determined from Maxwell's circulation equations.

While the scalar wave equation is separable in spherical coordinates, it is not obvious that a scalar solution is of any value in the determination of a vector field.

There is no loss of generality in the assumption of a time harmonic solution to eq (1); ie., $E = E(x_1, x_2, x_3)\, e^{-j\omega t}$, by which eq (1) becomes

$$\left[\nabla^2 - \left(\frac{\omega}{c}\right)^2\right] E = 0 \tag{2}$$

the so-called 'Helmholtz equation', which may be viewed as sort of Fourier transform of the wave equation. Then, the characteristic value (ω/c) is determined by boundary conditions on the spatial solution for the vector E. A complete, persuasive solution of the vector wave equation in spherical coordinates cannot be demonstrated briefly, but a résumé of the solution is appropriate as that is the subject of this paper.

For the axially symmetric case ($\partial/\partial\varphi = 0$) Bromwich(13) has shown that the wave equation separates completely into two sets, TE(H_r, H_ϑ, E_φ) and TM (E_r, E_ϑ, H_φ), that is, resonances having either radial magnetic or electric components. In this case it is only necessary to solve the circulation equations to completely define the field.

Alternatively, Shelkunoff(9) has shown that the general solution of the wave equation in spherical coordinates results in three sorts of waves, one with the magnetic field normal to the ray, or radius of propagation, (TM), one with the electric field normal to the ray, (TE), and one with both normal to the ray, and to each other, (TEM); a spherical boundary of course eliminates the TEM solution so that fortunately, perhaps, only two cases exist physically in a cavity ($H_r = 0$ or $E_r = 0$).

A technique of solving the spherical vector wave equation, assuming that either H_r or E_r vanishes, is to replace the vector with a potential or stream function by which means the wave equation can be reduced to a separable scalar wave equation, the solution of which is

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left(k - \frac{m(m+1)}{r^2}\right) = 0 \tag{3}$$

$$\frac{d^2\Theta}{d\vartheta^2} + \cot\vartheta\,\frac{d\Theta}{d\vartheta} + m(m+1) - \frac{m}{\sin^2\vartheta} = 0 \tag{4}$$

$$\frac{d^2\Phi}{d\varphi^2} + m\,\Phi = 0 \tag{5}$$

In the sphere periodicity requires eq (5) to have the solution ($m = n^2$),

$$\Phi = \cos n\varphi \qquad \text{(n an integer)} \tag{6}$$

Eq (4) is Legendre's equation, the solution of which is

$$\Theta = P_m^n(\cos\vartheta) \qquad m = 0, 1, 2, \ldots\ n \le m \tag{7}$$

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Eq. (3) has solutions in Bessel functions,

$$R = \frac{1}{\sqrt{kr}}\, J_{m+\frac{1}{2}}(kr) \tag{8}$$

The constant k is determined from the boundary conditions; for TE modes ($E_r = o$)

$$J_{m+\frac{1}{2}}(kr) = 0 \tag{9}$$

and for TM modes ($H_r = o$)

$$\frac{d}{d\,kr}\left[\sqrt{kr}\; J_{m+\frac{1}{2}}(kr)\right] = 0 \tag{10}$$

Of course the arguments kr are the discrete modes of resonance.

There has come into use in recent years a convenient notation,

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\; J_{n+\frac{1}{2}}(x) \tag{11}$$

having a recursion relation[12]

$$\begin{aligned} j_n(x) &= f_n(x)\sin x + (-1)^{n+1} f_{-(n+1)}(x)\cos x \\ f_{n-1}(x) + f_{n+1}(x) &= (2n+1)\, f_n(x)/x \\ f_0(x) = 1/x \qquad & f(x) = 1/x^2 \end{aligned} \tag{12}$$

from which it may be seen that:

$$\begin{aligned} j_0(x) &= \frac{\sin x}{x} \\ j_1(x) &= \frac{\sin x}{x} - \frac{\cos x}{x} \\ j_2(x) &= \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x \\ j_3(x) &= \left(\frac{15}{x^4} - \frac{6}{x^2}\right)\sin x - \left(\frac{15}{x^3} + \frac{1}{x}\right)\cos x \end{aligned} \tag{13}$$

Zeroes are therefore given by

$$\begin{aligned} \sin x_{0n} &= 0 \\ \tan x_{1n} &= x_{1n} \\ \tan x_{2n} &= \frac{3x_{2n}}{3 - x_{2n}^2} \\ \tan x_{3n} &= \frac{x_{3n}(15 + x_{3n}^2)}{(15 - 6x_{3n}^2)} \end{aligned} \tag{14}$$

These zeroes of eqs. (9) and (10) are listed in Table I & II[12]

TABLE I

$J_{m+\frac{1}{2}}(X_n) = 0$ $TE_{m,n}$ Modes

m\n	1	2	3	4	5	6	7
1	4.4934	7.7253	10.9041	14.0662	17.2208	20.3713	23.5195
2	5.7635	9.0950	12.3229	15.5146	18.6890	21.8539	
3	6.9879	10.4171	13.6980	16.9236	20.1218	23.3042	
4	8.1826	11.7049	15.0397	18.3013	21.5254	24.7276	
5	9.3558	12.9665	16.3547	19.6532	22.9046		
6	10.5128	14.2074	17.6480	20.9835	24.2628		
7	11.6570	15.4313	18.9230	22.2953			
8	12.7908	16.6410	20.1825	23.5913			
9	13.9158	14.8386	21.4285	24.8732			
10	15.0335	19.0259	22.6627				
11	16.1447	20.2039	23.8865				
12	17.2505	21.3740					
13	18.3513	22.5368					
14	19.4477	23.6932					
15	20.5402	24.8438					
16	21.6292						
17	22.7150						
18	23.7978						
19	24.8780						

In the general theory of cavity resonances it has been observed that resonance is given by

$$\left(\frac{\omega}{c}\right)^2 = \left(\frac{A}{a}\right)^2 + \left(\frac{B}{b}\right)^2 + \left(\frac{D}{d}\right)^2 \tag{16}$$

where a, b and d are transverse dimensions, and A, B and D are constants appropriate to the cavity geometry and nature of the electromagnetic mode. As an example, for the TM-111 mode in a rectangular box eq.(16) becomes

$$\left(\frac{\omega}{c}\right)^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2 \tag{17}$$

where the transverse dimensions are a and d and the length L.

When resonance does not depend on one of the dimensions (as it does not, eg. in those modes with zero indices) the applicable numerator of the indifferent dimension vanishes, as for the TE-101 mode in the above box, where $D = 0$ and

$$\left(\frac{\omega}{c}\right)^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{\pi}{b}\right)^2 \tag{18}$$

In addition, in some cases the transverse dimensions are indistinguishable (as in the right circular cylinder),

$$\left(\frac{\omega}{c}\right)^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{p_{mn}}{r}\right)^2 \tag{19}$$

L and r being the length and radius, n an integer and p_{mn} the n-th root of the m-th order Bessel function of the first kind (or its derivative).

In the spherical resonator it is doubtless obvious that a supposable efflourescence of indices will not occur because the geometry implies multiple degeneracy.

On the other hand, this degeneracy engenders a peculiar situation; any particular mode, eg. $TX_{m,n,p}$, which arises as the n-th root of the m-th order spherical Bessel function, has $n+1$ degeneracies ($0 \le p \le n$), all of which have the same frequency (and, curiously enough, the same Q), although very different field configurations.

TABLE II

$$\frac{d}{dx}\left[\sqrt{x}\; J_{m+\frac{1}{2}}(x)\right] = 0 \qquad TM_{mn} \text{ modes}$$

m\n	1	2	3	4	5	6	7
1	2.7437	6.1168	9.3166	12.4859	15.6439	18.7963	21.9455
2	3.8202	7.4431	10.7130	13.9205	17.1027	20.2720	
3	4.9734	8.7218	12.0636	15.3136	18.5242	21.7139	
4	6.0620	9.9675	13.3801	16.6742	19.9154		
5	7.1402	11.1890	14.6701	18.0085	21.2815		
6	8.2109	12.3915	15.9387	19.3212			
7	9.2755	13.5787	17.1896	20.6154			
8	10.3353	14.7534	18.4255	21.8939			
9	11.3910	15.9174	19.6485				
10	12.4434	17.0723	20.8603				
11	13.4929	18.2193					
12	14.5398	19.3593					
13	15.5845	20.4932					
14	16.6272	21.6216					
15	17.6682						
16	18.7076						
17	19.7455						
18	20.7821						
19	21.8175						

Despite the opening remark of this paper it has recently been proposed at CERN-LEP to implement the Stanford SLED energy storage scheme using a spherical cavity[10]. The intent included provision for tuning by means of perturbation, which presumably did not "work" well. The diameter of a sphere is determined by the temperature of the material,

$$\frac{da}{dT} = \alpha a \tag{20}$$

α being the coefficient of thermal expansion (16 ppm per deg. C for copper). Therefore, the temperature tuning range is given by

$$\frac{d\omega}{\omega} = -\alpha\, dT \tag{21}$$

which indicates that temperature regulation of the cavity would provide adequate tuning range, though of slow response time.

There is a widely known rule that the number of resonances (N) of a cavity of volume V having wavelengths greater than a specified value (λ_0) is of the order[11]

$$N = \frac{8\pi}{3}\,\frac{V}{\lambda_0^3} \tag{22}$$

For a spherical cavity, noting that $r/\lambda = p_{mn}/2\pi$,

$$N = \left(\frac{2}{3\pi}\right)^2 p_{mn}^3 \tag{23}$$

Table III presents a count of resonances (including degeneracies.

TABLE III

p_{mn}	N(eq 22)	N(Tables I & II)
5	6	8
10	45	46
15	152	131
20	360	290

(1) J. J. Thompson, Notes on Recent Researches in Electricity and Magnetism (1893); G. Mie, Annalen der Physik, 25,377 (1908)

(2) These remarks are taken from a lecture by H. Rothe, at the Institute of Technology, Karlsruhe printed in Elektrotechnische Zeitschrift 78,247 (April, 1957). That the remarks are a justifiable assessment may be inferred from the view of H. Hertz, Untersuchungen uber die Ausbrietung der Electrischen Kraft (1892) p6 to the effect that "in 1879 Maxwell's equations were not generally accepted in Germany."

(3) P. Debye, Ann. d Phys. 30,57 (1909)
T. Bromwich, Phil. Mag. (6) 38, 143 (1919)
F. Borgnis, Ann. d. Phys. (5) 35, 359 (1939)

(4) J. Stratton, Electromagnetic Theory, p 560 (1941); S. Schelkunoff, Electromagnetic Waves, p294 (1943); W. Smythe, Static and Dynamic Electricity, p537 (1950); P. Morse and H. Feshbach, Methods of Theoretical Physics, ii, 1870 (1953); W. Panofsky and M. Phillips, Classical Electricity and Magnetism, p201 (1955); G. Gobau, Electromagnetische Wellenleiter und Hohlraume, (1955) USAEC Transl. Electromagnetic Waveguides and Cavities, p237 (1961); J. Jackson, Classical Electrodynamics, p538 (1962)

(5) P. Staekl, Comptes Rendu 116, 485 (1893); Comptes Rendu 121, 489 (1895) A rather complete discussion of Staekl's determinant is given in Morse and Fesh-Bach, Meth. Theor. Phys., i, 509. See also H. Robertson, Math. Annalen 98, 749 (1927) L. Eisenhart, Annals of Math 35, 284 (1934)

(6) LALA, H. Hoyt et al., Rev. Sci. Instr., 37, 755 (1966); SUPERFISH, K. Halbach, Part. Accelerators, 7, 213 (1976); QLFISH, S. Okomura, Proc. 1981 Lin. Acc. Conf. LASL, LA-9234-C, p200; URMEL, T. Wieland, DESY 83-005 (1983)

(7) S. Ramo and J. Whinnery, Fields and Waves, p399 (1944)

(8) An intuitive argument of this sort was given by W. Hansen, "A Type of Electrical Resonator," JAP 9, 654 (1938)

(9) S. Schelkunoff, "Transmission Theory of Spherical Waves," Trans. AIEE, 57, 774 (1938)

(10) A Fiebig and R. Hohbach, IEEE Trans. Nuc. Sci., NS-30, 3563 (1983) See also H. Henke, "Spherical Modes", CERN-ISR-RF/81-29 (1981)

(11) G. Roe, "Frequency Distribution of Normal Modes," JASA 13, 1 (1941)

(12) NBS Handbook of Mathematical Functions, Applied Mathematics Series No. 55 (1964) p467 Eds. M. Abramowitz and I. Stegun; E. Butkov, Mathematical Physics (1968) p381; Janke and Emde, Table of Functions (1945 ed) p154