

ANALYTICAL METHODS OF ELECTRODE DESIGN FOR A
RELATIVISTIC ELECTRON GUN*

G. J. Caporaso, A. G. Cole and J. K. Boyd
Lawrence Livermore National Laboratory
P.O. Box 808
Livermore, California 94550

Abstract

The standard paraxial ray equation method [1] for the design of electrodes for an electrostatically focused gun is extended to include relativistic effects and the effects of the beam's azimuthal magnetic field. Solutions for parallel and converging beams are obtained and the predicted currents are compared against those measured on the High Brightness Test Stand [2].

Introduction

A well known technique for designing electrostatically focused guns makes use of the paraxial ray equation [1,3]. Essentially, a differential equation of motion is derived which gives the behavior of the beam envelope in terms of the second derivative of the potential with respect to the longitudinal coordinate. The desired behavior of the beam envelope in the longitudinal direction is prescribed and the equation is used to solve for the potential distribution consistent with that envelope specification. The solution is then used to set the potentials of the accelerating electrodes and their spacings. This technique is extended in the present work to include effects due to the beam's azimuthal magnetic field and relativity.

Method of Analysis

We consider an azimuthally symmetric system and consider only static fields. Additionally, we neglect any azimuthal component of velocity of the beam electrons. Thus, we may write a complete set of equations which govern the motion of the electrons

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \quad (1)$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi\vec{J}}{c} \quad (2)$$

$$\frac{d}{dt} (\gamma m \vec{v}) = -e(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) \quad (3)$$

$$\vec{E} = -\vec{\nabla}\phi \quad (4)$$

$$(\gamma - 1) mc^2 = e\phi \quad (5)$$

Equations (1) and (4) yield

$$\frac{1}{r} \frac{\partial}{\partial r} (rE_r) = 4\pi\rho + \frac{\partial^2 \phi}{\partial z^2} \quad (6)$$

while Eq. (2) yields $B_\theta = 2I_r/cr$ where I_r , the enclosed current at radius r is given by

$$I_r = \int_0^r 2\pi r' dr' J_z(r')$$

and $J_z = \rho v_z = \rho c \beta_z$, the z -component of the current density. Setting $dt = dz/v_z = dz/c\beta_z$ the radial component of Eq. (3) can be written as

$$mc^2 \beta_z \frac{\partial}{\partial z} (\beta_z \gamma \frac{\partial r}{\partial z}) = -e(E_r - \beta_z B_\theta) \quad (7)$$

We now introduce the paraxial approximation which requires that $\beta_r \ll \beta_z$ so that $\beta \approx \beta_z$. If in addition, $\gamma^2 \beta_r^2 \ll 1$ then $\gamma^2 \approx (1 - \beta_z^2)^{-1}$ and Eq. (7) may be written as

$$mc^2 \beta \frac{\partial}{\partial z} (\gamma \beta \frac{\partial r}{\partial z}) = -e(E_r - \beta B_\theta) \quad (8)$$

We now make the further simplifying assumption that the beam profile is flat, i.e. $\rho(r) = \text{constant}$. We also approximate $\beta(r)$ and $\partial^2 \phi / \partial z^2$ by their values at $r = 0$. With these assumptions E_r may be evaluated from Eq. (6) as

$$E_r = \frac{2\pi r J_z}{\beta c} + \frac{r\phi''}{2}$$

where the prime denotes $\partial/\partial z$. We may also find I_r and hence B_θ . Using these results, Eq. (5) and the fact that $\beta\gamma = \sqrt{\gamma^2 - 1}$ we may rewrite Eq. (8) as

$$\frac{\sqrt{\gamma^2 - 1}}{\gamma} \frac{\partial}{\partial z} \left[\sqrt{\gamma^2 - 1} \frac{\partial r}{\partial z} \right] = -r \left[\frac{2\pi J_z}{I_0 \gamma \sqrt{\gamma^2 - 1}} + \frac{\gamma''}{2} \right] \quad (9)$$

where $I_0 = mc^3/e \approx 17 \text{ kA}$.

Parallel Beam

We now consider the case of a parallel beam where $\partial r / \partial z = 0$. Equation (9) then immediately simplifies to

$$\gamma'' + \frac{4\pi J_z}{I_0 \gamma \sqrt{\gamma^2 - 1}} = 0$$

If L is the anode-cathode gap distance then we may define a new independent variable ζ as z/L . Then we may write the above equation as

$$\gamma'' - \frac{\alpha}{\sqrt{\gamma^2 - 1}} = 0$$

where $\alpha = -4\pi J_z L^2 / I_0$ and where a prime now denotes differentiation with respect to ζ . A first integral of this equation is

$$\frac{1}{2} \left(\frac{d\gamma}{d\zeta} \right)^2 - \alpha \sec^{-1} = C$$

If we now impose boundary conditions appropriate for space charge limited flow, namely $\gamma(0) = 1, \gamma'(0) = 0$ and $\gamma(1) = \gamma_0$ we have

* Performed jointly under the auspices of the U.S. DOE by LLNL under W-7405-ENG-48 and for the DOD under DARPA, ARPA Order No. 4395, monitored by NSWC.

$$\int_1^\gamma \frac{d\gamma}{\sqrt{\sec^{-1}\gamma}} = \sqrt{2\alpha} \zeta \tag{10}$$

where α is given by Eq. (10) evaluated at $\zeta = 1$ with $\gamma = \gamma_0$. γ vs. ζ is plotted in Fig. 1. With α determined J_z is immediately given as

$$J_z = -\frac{I_0 \alpha}{4\pi L^2} \tag{11}$$

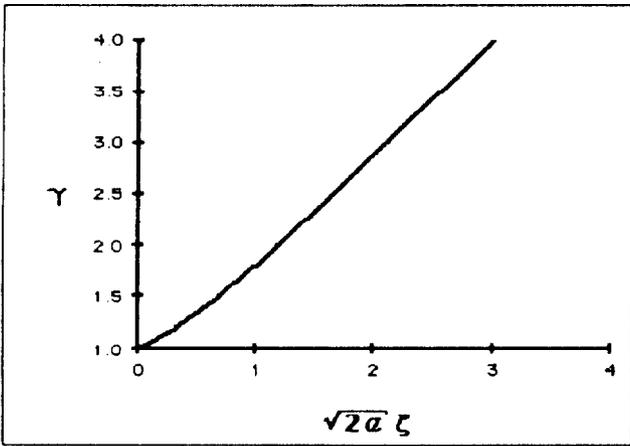


Fig. 1. Solution to Eq. (10). γ is plotted vs. $\sqrt{2\alpha} \zeta$.

Exponentially Converging Beam

If $r(s,z) = r_0(s)e^{-\alpha z}$ where s labels the radius at the cathode then we have $v_r = dr/dt = -\alpha r_0 e^{-\alpha z} dz/dt = -\alpha r v_z$. Thus, $J_r = \rho v_r = -\alpha r J_z$. Again, we take ρ and β to be their on-axis values. Thus, $J_z = J_z(z)$ and $\vec{\nabla} \cdot \vec{J} = 0$ gives

$$\frac{1}{r} \frac{\partial}{\partial r} (r J_r) + \frac{\partial J_z}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} (-\alpha r^2 J_z) + \frac{\partial J_z}{\partial z} = 0$$

with the solution $J_z = J_0 e^{2\alpha z}$.

Equation (9) now becomes

$$\gamma'' + \frac{4\pi J_0 e^{2\alpha z} - 2\alpha e^{\alpha z} \sqrt{\gamma^2 - 1}}{I_0 \gamma \sqrt{\gamma^2 - 1}} \frac{\partial}{\partial z} (\sqrt{\gamma^2 - 1} e^{-\alpha z}) = 0$$

If we define $\zeta \equiv \alpha z$ and $\lambda \equiv 4\pi J_0 / I_0 \alpha^2$ we have finally that

$$\gamma'' - 2\gamma' + \frac{2(\gamma^2 - 1)}{\gamma} + \frac{\lambda e^{2\zeta}}{\gamma \sqrt{\gamma^2 - 1}} = 0 \tag{12}$$

Again, we choose boundary conditions appropriate to space charge limited flow: $\gamma(0) = 1$, $\gamma'(0) = 0$ and $\gamma(\zeta_0) = \gamma_0$ where a prime now denotes differentiation with respect to ζ . Equation (12) along with its three boundary conditions is an eigenvalue problem for the parameter λ . A value of λ is assumed and Eq. (12) is integrated from $\zeta = 0$ to $\zeta = \zeta_0$. If the boundary condition $\gamma = \gamma_0$ is not

satisfied at $\zeta = \zeta_0$ then a new value of λ is chosen and the entire procedure repeated until the unique value of λ is found that permits the satisfaction of all three boundary conditions. J_0 is then found from the relation

$$J_0 = \frac{I_0 \alpha^2 \lambda}{4\pi} \tag{13}$$

A plot of the solution to Eq. (12) is given in Fig. 2.

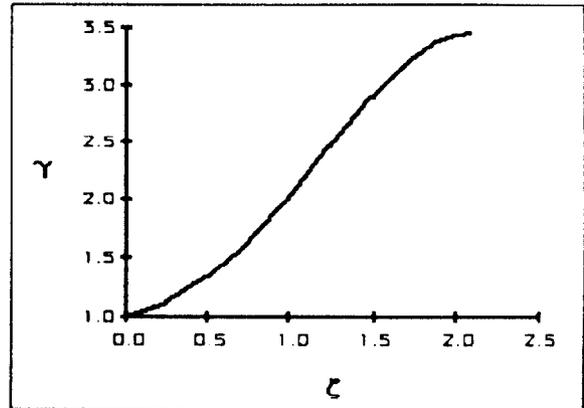


Fig. 2. Solution to Eq. (12) corresponding to the values used as a design example in the text. The solution here is for the additional boundary condition $\gamma'(\zeta_0) = 0$ which essentially determines ζ_0 and hence the parameter α . γ vs. ζ is plotted.

Design Examples

Two examples corresponding to different designs of the electron gun of the High Brightness Test Stand are now discussed [2,4]. The first example is the design of a pentode structure for a parallel beam. The four voltage differences between the five electrodes are required to be equal. The problem then reduces to solving Eq. (10) for the location of the four equally graded accelerating electrodes with respect to the cathode.

The design used a cathode diameter of 1 - 1/8 inches (1.43 cm radius) and an anode-cathode gap of 1.95 inches (4.95 cm) with a peak anode voltage of 1.25 MV. From Fig. 1 α is found to be 3.21. Using Eq. (11) to obtain the current density, the calculated beam current $\pi r_c^2 J_z$, is found to be 1.18 kA as compared to the experimentally determined value of ~ 1.25 kA. The interval $\gamma_0 - 1$ is divided into four equal parts corresponding to the four accelerating electrodes and the positions of those electrodes read from Fig. 1. With the value of α known the actual locations of the electrodes are determined as: cathode at $z = 0$, first electrode at $z = 1.54$ cm, 2nd electrode at 2.77 cm, third electrode at $z = 3.82$ cm and anode at $z = 4.85$ cm.

The second example is that of a converging beam produced by a simple diode configuration. The cathode used in this design is a concave thermionic emitter of radius 4.46 cm. The axial anode-cathode gap is 15 cm and the peak anode voltage is 1.25 MV.

An additional requirement was imposed on the solution of Eq. (12) for this design; namely that $\gamma' = 0$ at the anode location. This requirement would result in a convergent beam at the anode with

no axial electric field present which presumably would be more easily transported by solenoids downstream of the anode. This fourth condition on the solution of Eq. (12) essentially determines α , the convergence parameter. This problem has a solution for the non-relativistic case when the self magnetic field of the beam is neglected [1]. The fully relativistic case including the self magnetic field also may be solved and is shown in Fig. 2. λ is determined to be $-.238$ and $\alpha = 2.1 \text{ cm}^{-1}$. The current density expected is found from Eq. (13). Multiplication by the cathode area gives 396 amperes as the expected diode current. The beam radius converges from an initial value at the cathode of 4.46 cm down to .546 cm at the anode. We find that r' at the edge of the cathode is .625 hence the paraxial approximation is not well satisfied. The diode was tested as designed with the exception that the curvature of the cathode was not in accord with the requirements of the solution to Eq. (12). Only ≈ 200 amps could actually be transported through the anode hole and the actual current emitted from the cathode could not be determined.

Summary

Using the paraxial approximation enabled the derivation of a differential equation for the applied potential in a relativistic electron gun. The equation also incorporates the effects of the beam's own azimuthal magnetic field. The resulting equation was considered for the case of a parallel beam and for the case of an exponentially converging beam. When the paraxial approximation is valid the solutions to the equation give reasonably accurate results in terms of expected current and axial placement of electrodes.

References

- [1] P. T. Kirstein, G. S. Kino, W. E. Waters, Space Charge Flow, New York: McGraw-Hill, 1967, Ch. 3, pp. 131-137.
- [2] D. L. Birx, "Technology of Magnetically Driven Accelerators," presented at the 1985 Particle Accelerator Conference, Vancouver, B.C., Canada, May 13-16, 1985.
- [3] Grivet, Electron Optics, Paris: Pergamon, 1964.
- [4] G. J. Caporaso and D. L. Birx, "Brightness Measurements on the Livermore High Brightness Test Stand," presented at the 1985 Particle Accelerator Conference, Vancouver, B.C., Canada, May 13-16, 1985.