BEAM BREAKUP WITH SMOOTH RAPID CURRENT BUILD-UP*

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Summary

Cumulative beam breakup in a high current linac can be represented by a set of difference equations for the Mth beam bunch in the Nth cavity. We here investigate the modification of the solution of these equations when the displaced beam current grows smoothly to its final value. Simulations show a dramatic reduction in the transient, even when the current growth takes place over only a few bunches. In our analysis, adiabatic results for large M and N are given for an exponential build-up of current corresponding to a "time constant" of T bunches. The solutions confirm the behavior observed in the simulations in all details, including the location of the transient peak, as well as the dramatic dependence of the magnitude of the transient on T.

Introduction

In cumulative beam breakup a transversely displaced bunched beam interacts with one or more transverse modes in a series of uncoupled identical cavities. As a result, the transverse beam displacement can grown by a large factor in successive cavities. This phenomenon has been analyzed by several authors.[1] In a recent publication[2] the difference equations for the Mth bunch in the Nth cavity have been solved exactly for the case of no acceleration. The magnitude, location and width of the transient growth (large M, N) obtained are in excellent agreement with numerical simulations using the difference equations.

In all these analyses the beam bunches are each assumed to contain the same charge. In the present paper we investigate the effect of a smooth build-up of the current, expecting little effect if the build-up is more rapid than the growth rate of the transient for a constant current. Suprisingly, there is a dramatic decrease in the magnitude of the transient even for a rapid current build-up.

Difference and Differential Equations

The difference equations for the transverse displacement $\xi(N,M)$ and angle $\theta(N,M)$ can be written as[2]

$$\xi(N+1,M) = M_{11}\xi(N,M) + \frac{\gamma_N M_{12}}{\gamma_{N+1}}\left[\theta(N,M) + \phi(N,M)\right] \quad (1)$$

$$\theta(N+1,M) + M_{21}\xi(N,M) + \frac{\gamma_N M_{22}}{\gamma_{N+1}}\left[\theta(N,M) + \phi(N,M)\right] \quad (2)$$

where

$$\phi(N,M) = \frac{1}{\gamma_N}\sum_{\ell=0}^{M-1} s_{M-\ell}\,\xi(N,\ell)\,R_{N\ell} \quad (3)$$

$$s_k = e^{-k\omega\tau/2Q}\sin k\omega\tau \quad (4)$$

Here the 2x2 matrix M_{ij} represents the transport and focussing between cavities, $\omega/2\pi$ and Q are the frequency and quality factor of the transverse cavity mode, τ is the time interval between bunch, γ_N is the particle energy at cavity N in units of mc^2 and $R_{N\ell}$ is a parameter proportional to the charge in the ℓth bunch as well as to the ratio of the shunt impedance to the Q of the Nth cavity.

Equations (1) and (2) have been solved exactly[2] for $R_{N\ell}$, M_{ij} and γ_N independent of N and ℓ. This does not appear to be possible for $R_{N\ell}$ dependent on ℓ and we therefore assume a smooth dependence and convert Eqs. (1) and (2) to differential equations in the absence of external focussing.

If we define $y(N,M)$ by $\xi(N,M)\,(\gamma_N/M12)^{1/2} = y(N,M)$ and assume that M_{12} and γ are slowly varying functions of N, we obtain

$$\frac{\partial^2 y(N,M)}{\partial N^2} = \sum_{\ell=0}^{M-1} P_{N\ell}\,s_{M-\ell}\,y(N,\ell) \quad (5)$$

where $P_{N\ell} \equiv \frac{R_{N\ell}M_{12}}{\gamma_N} = p(\ell)\,q(N)$ is assumed to factor into independent functions of M and ℓ, with $p(\ell)$ normalized such that $p(\infty) = 1$.

The steady state value of y clearly satisfies

$$\frac{\partial^2 y(N,\infty)}{qN^2} = q(N)\,y(N,\infty)\sum_{k=1}^{\infty} s_k \quad (6)$$

so that we can write for the approach to equilibrium

$$\frac{\partial^2 Y(N,M)}{\partial N^2} = q(N)\sum_{\ell-0}^{M-1} p(\ell)s_{M-\ell}\,Y(N,\ell) + q(N)y(N,\infty)\,\mathrm{Im}[He^{iM\alpha}] \quad (7)$$

where $Y(N,M) \equiv y(N,M) - y(N,\infty)$, $\alpha = \omega\tau(1 + i/2Q)$, and

$$H = e^{-iM\alpha}\left[\sum_{k=1}^{\infty} e^{ik\alpha} - \sum_{k=1}^{M} p(M-k)\,e^{ik\alpha}\right]. \quad (8)$$

If we choose an exponential transient for the current of the form

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$$p(\ell) = 1 - e^{-\ell/T} \quad (9)$$

where T is the "time constant" for the build-up, we find for $M \gg T$

$$H(T) = \frac{1}{e^{-i\alpha} - 1} - \frac{1}{e^{-i\alpha} - e^{-1/T}}$$

Clearly $|H(T)| \approx K$ for $T = 0$, and K^2/T for $T \gg 1$, where $K^{-1} \approx 2|\sin(\omega\tau/2)|$, as long as we are not at a resonance $\omega\tau \approx 2n\pi$.

Let us now set

$$Y(N,M) = [u(N,M)e^{-iM\alpha} + u^*(N,M)e^{iM\alpha}]/2 \quad (10)$$

and neglect all terms whose phases vary rapidly with M. [We assume u(N,M) varies more slowly with M than $\exp(iM\alpha)$.] This leads to

$$\frac{\partial^2 u(N,M)}{\partial N^2} = \frac{i}{2} v(N,M), \quad \frac{\partial v(N,M)}{\partial M} = q(N)p(M)u(N,M) \quad \begin{matrix}(11)\\(12)\end{matrix}$$

where

$$v(N,M) = q(N) \left[\sum_{\ell=0}^{M-1} p(\ell)u(N,\ell) - y(N,\infty)H(T) \right] \quad (13)$$

Transient Solution for Approach to Equilibrium

We can solve Eqs. (11) and (12) approximately by trying a solution of the form

$$\begin{Bmatrix} u(N,M) \\ v(N,M) \end{Bmatrix} \approx \begin{Bmatrix} A \\ B \end{Bmatrix} \exp[f(M)g(N)] \quad (14)$$

where the dominant M and N dependence is included in f(M) and g(N). Equations (11) and (12) then become

$$A[f(M)]^2[g'(N)]^2 \approx iB/2, \quad Bf'(M)g(N) \approx Ap(M)q(N) \quad \begin{matrix}(15)\\(16)\end{matrix}$$

requiring

$$3[f(M)]^2 f'(M) \approx p(M), \quad g(N)[g'*N)]^2 \approx 3iq(N)/2 \quad \begin{matrix}(17)\\(18)\end{matrix}$$

These equations can be solved to give

$$f(M) \approx \left[\int_0^M p(\ell)d\ell\right]^{1/3} \quad (19)$$

$$g(N) \approx \frac{3}{2} \exp\left(\frac{i\pi}{6}\right) \left[\int_0^N [q(n)]^{1/2} dn\right]^{2/3} \quad (20)$$

We now need to estimate the normalization factors A and B. Since $f(0) = 0$ according to Eq. (19), we find from Eqs. (13) and (14)

$$B \approx v(N,0) = -2q(N)\, y(N,\infty)\, H(T). \quad (21)$$

We now assume that Eq. (21) gives the dominant dependence of both A and B on T, namely proportionality to H(T). But for $T = 0$ our previous exact solution[2] holds, so that we can modify Eq. (20) in Reference 2 by the factor $|H(T)/H(0)|$ to write as our final result for the approach to equilibrium

$$|\xi(N,M) - \xi(N,\infty)|/\xi_o \approx$$

$$\left|\frac{H(T)}{H(0)}\right| \frac{K\sqrt{2}}{\sqrt{\pi}\, 3^{5/4}} \frac{\sqrt{G}}{(M-T)^{5/6}} \exp\left(-\frac{M\omega\tau}{2Q} + G(M-T)^{1/3}\right) \quad (22)$$

where

$$G(N) = \mathrm{Re}[g(N)] = \frac{3\sqrt{3}}{4} \left[\int_0^N [q(N)]^{1/2} dn\right]^{2/3} \quad (23)$$

and where we have used the approximation $f(M) \approx (M-T)^{1/3}$ for the exponential build-up in Eq. (9) in the region $1 \ll T \ll M$.

Numerical Results

Simulations have been performed for the parameters in Reference 2, namely

$$\frac{\omega\pi}{2\pi} = 1.846, \quad \frac{\omega\tau}{2Q} = 5.80 \times 10^{-3},$$

$$q(N) = 2.88 \times 10^{-3}, \quad \xi_o = 1\text{mm}.$$

for T = 0,5,20,80, and the results are shown in Figures 1-4 for N = 30. If we define

$$w(M) = \ln \frac{|\xi(N,M) - \xi(N,\infty)|}{\xi_o} + \frac{M\omega\tau}{2Q} + \frac{5}{6}\ln(M-T) \quad (24)$$

and plot w(M) vs $(M-T)^{1/3}$, Eq. (22) predicts that the result will be a straight line with slope G(N) and w intercept $\ln[|H(T)|\sqrt{G}\,(4/3)^{1/4}/3\sqrt{\pi}]$. Figure 5 shows that the simulations lead to almost perfect straight lines, with intercepts and slopes as given in the table:

	Simulation		Predicted	
T	Intercept	Slope	Intercept	Slope
0	-1.32	1.77	-1.24	1.78
5	-2.89	1.77	-2.78	1.78
20	-4.18	1.77	-4.16	1.78
80	-5.57	1.77	-5.55	1.78

We clearly have an excellent analytical model for the transient approach to the steady state for a current pulse which builds up exponentially. It confirms the rapid decrease of the maximum amplitude with T as contained in the factor $|H(T)/H(0)| \approx K/T$.

Non-Transient Behavior

The simulations also show rather simple behavior in the region $M < T$ (see for example Fig. 4 for T = 80). We can derive the result for this range of parameters for the case constant $q(N) = q$ by starting with Eq. (5) for y(N,M) or $\xi(N,M)$. Specifically, we write $\xi(N,M)$ as a power series in $N(\gg 1)$, and eventually obtain

$$\xi^{(1)}(N,M) \approx \xi_o \cosh N\sigma_M \quad (25)$$

where $\cosh \sigma_M \approx 1 + \frac{\sigma_M^2}{2} = 1 + \frac{qp(M)\cot(\omega\tau/2)}{4}$.

Remarkably, Eq. (25) agrees completely with the non-resonant steady state result in Eqs. (57)-(59) of Reference 2, and thus describes the non-transient behavior for all relative values of M and T. In fact, one can write a general result for $\xi(N,M)$ for all M by replacing $\xi(N,\infty)$ by $\xi^{(1)}(N,M)$ in Eq. (22). The solid line in Fig. 4, representing $\xi^{(1)}(N,M)$ for N = 30, T = 80, shows the symmetric behavior of the transient with respect to $\xi^{(1)}(N,M)$.

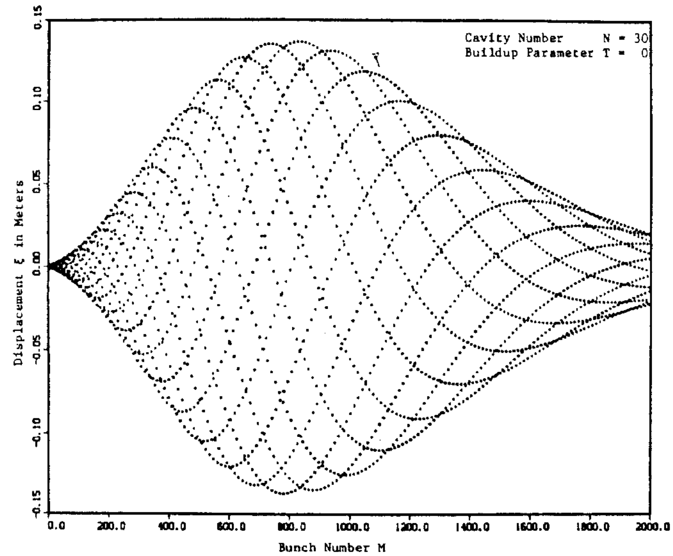

Fig. 1 $\xi(N,M)$ vs M for Current Build-up with T = 0

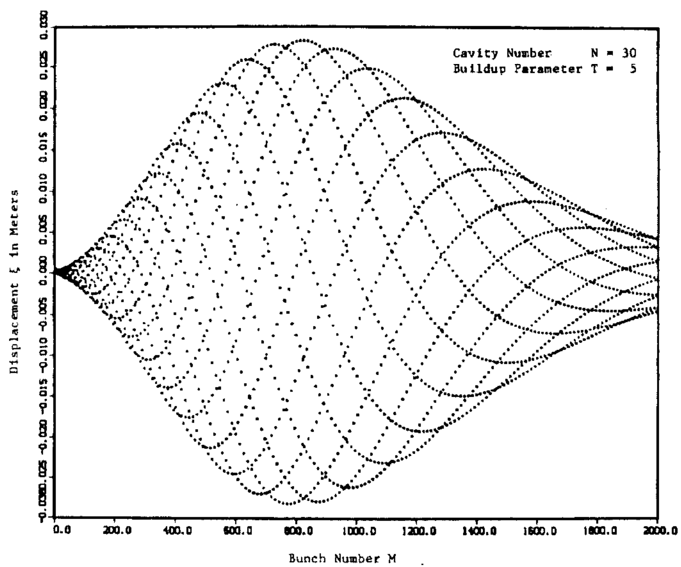

Fig. 2 $\xi(N,M)$ vs M for Current Build-up with T = 5

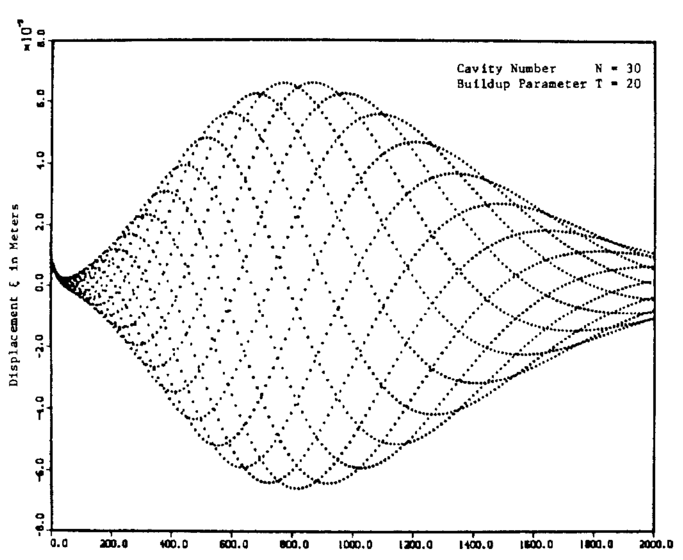

Fig. 3 $\xi(N,M)$ vs M for Current Build-up with T = 20

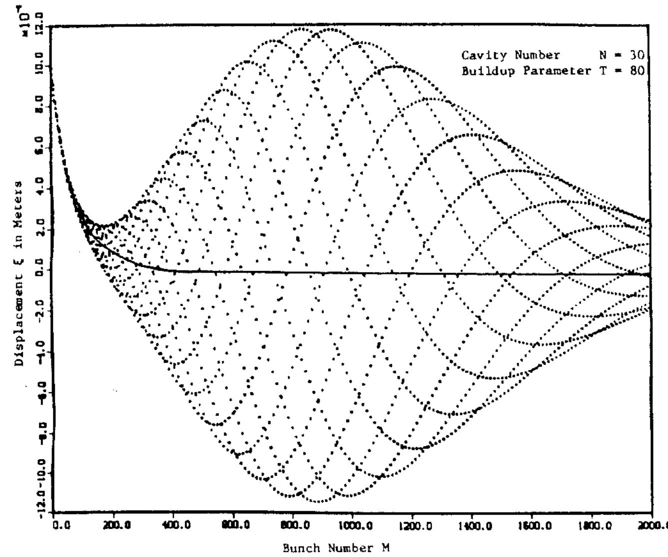

Fig. 4 $\xi(N,M)$ vs M for Current Build-up with T = 80

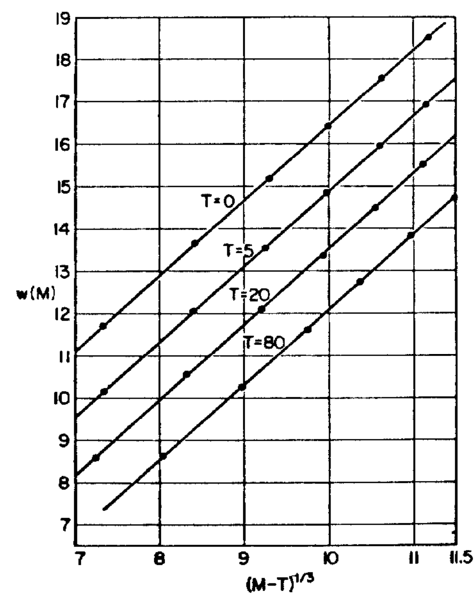

Fig. 5 Plot of w(M) vs $(M-T)^{1/3}$ for T = 0,5,20,80

References

1. See for example P.B. Wilson, Proceedings of the 1981 Accelerator Summer School, Fermilab, p. 450; R. Helen and G. Loew, Linear Accelerators, edited by P.M. Lapostolle and A.L. Septier, John Wiley and Sons, 1970, p. 201; Neil, Hall and Cooper, Particle Accelerators 9, 213 (1979).

2. Gluckstern, Cooper and Channell, Particle Accelerators 16, 125 (1985).