BEAM BREAKUP WITH RANDOM INITIAL DISPLACEMENT*

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Summary

We have obtained analytic results for the displacement which results from cumulative beam breakup with random initial displacement of each beam bunch. The results are in excellent agreement with simulations, and confirm an enhancement of the single pulse maximum by a factor of the order of the square root of the "width" of the single pulse transient, as expected.

Introduction

In a previous paper[1] we have solved the difference equations for cumulative beam breakup exactly for the case of no acceleration. Explicit expressions were obtained for the steady state solution in the case of a steady or modulated transverse initial displacement. In addition, the transient solution was obtained for a single initial displaced bunch, as well as for the approach to the steady state.

Our results show a significant enhancement of the transverse deflection if the initial bunch displacement is modulated at a frequency which in resonance with the sum or difference of the transverse mode frequency and any multiple of the beam bunch frequency. Since a beam with random initial beam displacement can be expected to have a portion of its spectrum in a resonant frequency region, we expect enhanced displacements in the rms case. This paper is an investigation of beam breakup in the case of random initial beam bunch displacements.

Analysis

The solution for the displacement of the Mth beam bunch in the Nth cavity for a single initial displacement, ξ_o, is[2]

$$\xi(N,M) = \xi_o[f(M) + f^*(M)] \tag{1}$$

with

$$f(M) \simeq \frac{\sqrt{F}}{2M\sqrt{6\pi}} e^{p(M) + iq(M)} \tag{2}$$

$$p(M) + iq(M) \equiv -\frac{M\omega\tau}{2Q} - iM\omega\tau + \frac{3}{4}(\sqrt{3} + i) F + \frac{i\pi}{12} \tag{3}$$

Here

$$F = (\hat{R} N^2 M)^{1/3} \tag{4}$$

where $\hat{R}$ depends linearly on the average beam current and the ratio of the transverse shunt impedance to the Q of each cavity. Considering only the M dependence in the exponent, one finds that $|\xi(N,M)|$ reaches a maximum

$$\frac{|\xi|_{max}}{\xi_o} = \frac{1}{\sqrt{6\pi}} \left(\frac{4}{3}\right)^{1/4} \frac{\sqrt{p_o}}{M_o} e^{p_o} \tag{5}$$

when

$$M = M_o = \left(\frac{3}{4}\right)^{3/4} N \left(\frac{Q}{\omega\tau}\right)^{3/2} (\hat{R})^{1/2} \tag{6}$$

where

$$p_o \equiv p(M_o) = M_o\omega\tau/Q \tag{7}$$

In the situation where all initial displacements are random, the displacement $\xi(N,M)$ can be obtained as a linear superposition of the solution for individual initial displacements. Specifically one has

$$\xi(N,M) \simeq \sum_{m=0}^{M} [f(m) + f^*(m)]\, \xi(0,M-m) \tag{8}$$

where $\xi(0,j)$ is the initial displacement of the jth bunch. The expected value of $\xi^2(N,M)$ as $M \to \infty$ for uncorrelated initial displacements is therefore

$$\langle\xi^2(N,\infty)\rangle \simeq 2\langle\xi_o^2\rangle \sum_{m=0}^{\infty} |f(m)|^2 \tag{9}$$

where we have neglected oscillatory terms and where $\langle\xi_o^2\rangle$ is the average square initial displacement.

It is clear from Eqs. (2) and (3) that the sum in Eq. (9) is dominated by the behavior of p(M) near $M = M_o$. If one considers only the m dependence in the exponent, and approximates the sum in Eq. (9) by an integral over m which is evaluated in a saddle point approximation, one finds

$$\frac{\langle\xi^2(N,\infty)\rangle}{\langle\xi_o^2\rangle} \simeq \frac{1}{6\sqrt{\pi}} \frac{\sqrt{p_o}}{M_o} e^{2p_o} \tag{10}$$

Clearly the order of magnitude of the result in Eq. (10) can be understood as the square of the maximum displacement in Eq. (5) multiplied by the "width" of

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the transient in Eq. (1), which is given approximately as

$$\Delta M \sim \sqrt{2\pi}\,[-p''(M_o)]^{-1/2} \sim 4M_o/\sqrt{p_o} \qquad (11)$$

Comparison with Simulations

Simulations have been performed with numerical values used in the previous paper[1], namely $R = 2.88 \times 10^{-3}$, $Q = 1000$, $\omega\tau = 11.6$, $N = 15{,}30$ and the results are shown in Figs. 1 and 2 for a random initial displacement whose distribution is uniform between -1 mm and 1 mm. For $N = 15$, one finds from Eqs. (6), (7), (10) that $M_o \simeq 520$, $p_o \simeq 6.0$, $\langle\xi^2(15,\infty)\rangle^{1/2} \simeq 6.4$ mm. For $N = 30$, one obtains $M_o \simeq 1040$, $p_o \simeq 12.0$, $\langle\xi^2(30,\infty)\rangle^{1/2} = 1.64$ m. If one assumes that the rms values in Figs. 1 and 2 are reduced from the peaks by a factor 2 because of oscillations of both the envelope and the displacement, one sees that the agreement between the simulation and the above prediction is quite good.

The striking pattern of the simulations, particularly the one in Fig. 2, can be understood by recognizing that the displacement for a given M can be thought of as the sum of the displacements with oscillatory phase in Eq. (8) from $m = M_o - \Delta M/2$ to $M_o + \Delta M/2$, where ΔM is the "width" of the single pulse transient. Clearly such a sum resembles a random walk phenomenon, and the points of small envelope in Figs. 1 and 2 correspond to the intervals in m for which the sum of the random contribution vanishes. Moreover, the envelope peaks are approximately separated by the value of ΔM in Eq. (11).

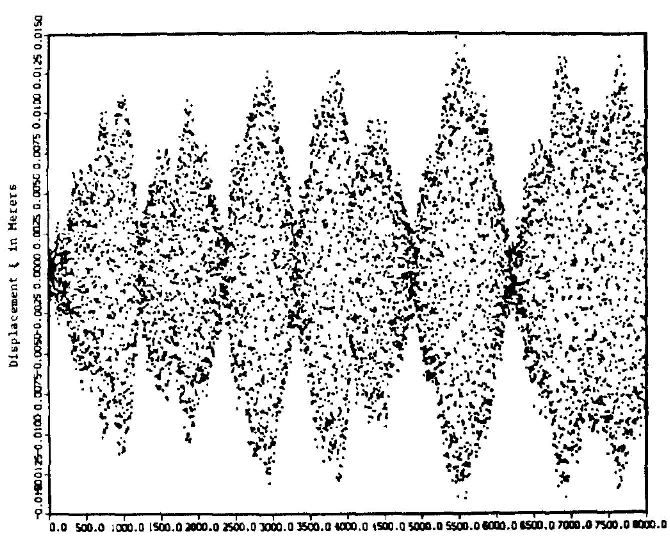

Fig. 1 $\xi(N,M)$ vs M for Cavity Number N = 15 with Random Initial Displacement

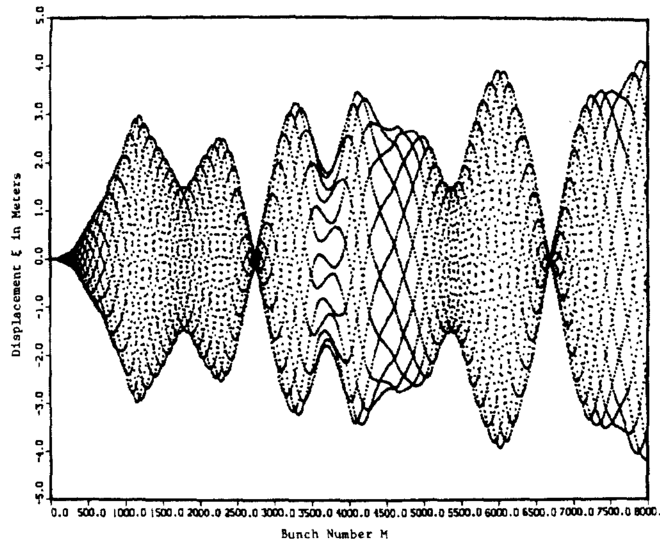

Fig. 2 $\xi(N,M)$ vs M for Cavity Number N = 30 with Random Initial Displacement

References

1. R.L. Gluckstern, R.K. Cooper and P.J. Channell, Particle Accelerators 16, 125 (1985).

2. loc. cit., Eqs. (69), (70).