

**THE EFFECT OF RADIATION DAMPING AND NOISE ON THE TRANSVERSE MODE COUPLING INSTABILITY DUE TO LOCALIZED STRUCTURES**

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**ABSTRACT**

The transverse mode coupling instability due to localized structures of a storage ring, like RF-cavities, is investigated in presence of radiation damping and quantum excitation for both synchrotron and betatron oscillations. Replacing the Vlasov equation by the Fokker-Planck equation, the longitudinal dynamics can only be described in terms of transition probabilities, whereas the transverse motion of the bunch barycenter remains deterministic. By a numerical solution of the associated integral equation, we obtain better estimates of the instability growth rates and we show the disappearance of the stop-bands associated with very high order dipole modes, which turn out to be damped proportionally to their mode number.

**1. INTRODUCTION**

A relativistic charged particle passing through the conducting structures of an accelerator induces electromagnetic wakefields which react on the particles following. This phenomenon gives rise to a collective force leading, under certain conditions, to a coherent single bunch instability generally described in terms of transverse mode coupling [1-2]. Indeed the particle transverse distribution can be decomposed into a series of orthogonal modes: each of these modes is characterized by a coherent frequency of oscillation which, for a given betatron tune  $\nu_z$  and synchrotron tune  $\nu_s$ , depends on the bunch current  $I_b$ . When two transverse modes happen to have the same frequency, they get strongly coupled and a slight increase of current causes one of them to become unstable.

Most analytical theories on beam stability [3] have made use of "distributed impedances", corresponding to a collective force which is smeared out all along the ring. As a result, they predict the existence of a threshold current  $I_{th}$  above which the combined effect of wakefields and longitudinal oscillations leads to a fast transverse blow-up of the bunch, characterized by a rise time comparable to the synchrotron period.

Wakefields are mainly generated near localized structures of a storage ring, like RF-cavities, bellows or other cross section variations of the vacuum chamber. Their global effect can often be represented by a transverse kick localized at a single point of the machine and such a model is commonly adopted in particle tracking by computer simulation [4].

In a previous paper [5] based on the Vlasov equation, we showed the existence of instability stop-bands at currents below threshold, which are due to the coupling between high order and low order modes. The stop-band pattern repeats periodically every half integer in the betatron tune  $\nu_z$  and, choosing  $\nu_z$  in the range  $[0, 1/2]$ , the bunch may become unstable at very low currents near the resonant values  $\nu_z = n \nu_s$  or  $\nu_z = 1/2 - n \nu_s$ . The maximum growth rate of the instability in a stop-band is roughly proportional to the width of the stop-band itself and decreases for increasing mode numbers. These predictions are in good agreement with the results of tracking and with the conclusions drawn from two-or-more particle models [2].

In an electron-positron storage ring, both longitudinal and transverse oscillations are affected by

radiation damping and quantum excitation: thus, in order to obtain more realistic results, the Vlasov equation has to be replaced by the Fokker-Planck equation [6-7]. Here we investigate the effects of damping and noise on the transverse mode coupling instability due to localized structures and show the disappearance of the stop-bands associated with very high order modes. By inspection of the imaginary part of the coherent frequencies, we can identify modes which become unstable above a given current and obtain better estimates of the instability growth rates.

Starting from the Fokker-Planck equation, in Section 2 we will derive an integral equation for the transverse dipole distribution function. In Section 3, this equation is reduced by a Fourier analysis to an equivalent eigenvalue problem which contains resonant coefficients, depending on the coherent frequency of the dipole modes. Section 4 concludes the paper by a discussion of the numerical results.

**2. THE FOKKER-PLANCK EQUATION**

We denote by  $(\tau, p_\tau)$  the synchrotron phase space and by  $(z, p_z)$  the normalized betatron phase space. At a fixed azimuth  $\theta_0$  along the machine, particles experience at each turn a transverse collective force  $F(\tau, t)$ , depending on their longitudinal position within the bunch. Taking into account radiation damping and quantum excitation [8], the single particle equations of motion can be written as follows:

$$\begin{aligned} \dot{\tau} &= p_\tau, & \dot{p}_\tau &= -\omega_s^2 \tau - 2\alpha_s p_\tau + 2(\alpha_s)^{1/2} \omega_s \sigma_s \xi_1(t), \\ \dot{z} &= p_z, & \dot{p}_z &= -\omega_z^2 z - 2\alpha_z p_z + 2(\alpha_z)^{1/2} \omega_z \sigma_z \xi_2(t) + \\ & & & + (\beta_z)^{1/2} \omega_z F(\tau, t) / \gamma m c. \end{aligned} \quad (1)$$

Here  $\beta_z$  is the amplitude function at  $\theta_0$ ,  $\gamma m c$  is the particle momentum in the extreme relativistic case,  $\alpha_s$  and  $\alpha_z$  are the synchrotron and betatron damping constants,  $\sigma_s$  is the rms bunch length (in time units),  $\sigma_z$  the rms transverse beam radius and  $\xi_1(t)$  and  $\xi_2(t)$  are two independent delta-correlated stochastic variables, describing the white noise associated with quantum excitation.

The system of stochastic equations (1) is equivalent to the following Fokker-Planck equation [6] for the phase space distribution function  $\psi(\tau, p_\tau, z, p_z, t)$

$$(\partial\psi/\partial t) = (L_s + L_z + M) \psi, \quad (2)$$

where  $L_s$  and  $L_z$  are two elliptic differential operators taking into account the effect of damping and noise on the synchrotron and betatron oscillations, respectively, whereas  $M$  is a first order differential operator associated with the transverse collective force:

$$\begin{aligned} L_s &= \omega_s^2 \tau (\partial/\partial p_\tau) - p_\tau (\partial/\partial \tau) + \\ & + 2\alpha_s (\partial/\partial p_\tau) [p_\tau + \omega_s^2 \sigma_s^2 (\partial/\partial p_\tau)], \\ L_z &= \omega_z^2 z (\partial/\partial p_z) - p_z (\partial/\partial z) + \\ & + 2\alpha_z (\partial/\partial p_z) [p_z + \omega_z^2 \sigma_z^2 (\partial/\partial p_z)], \\ M &= - [(\beta_z)^{1/2} \omega_z F(\tau, t) / \gamma m c] (\partial/\partial p_z). \end{aligned} \quad (3)$$

In order to specify the form of the collective force  $F(\tau, t)$  and to transform the Fokker-Planck equation (2) into an integral equation, we introduce the transverse dipole distribution  $D(\tau, p_\tau, t)$  and the associated transverse momentum distribution  $P(\tau, p_\tau, t)$ . They are first order moments of  $\psi$  with respect to  $z$  and  $p_z$ . The usual dipole density  $\bar{D}(\tau, t)$  is obtained by integrating  $D(\tau, p_\tau, t)$  over  $p_\tau$  and the collective force corresponding to a single localized structure can then be written as in [5]

$$F(\tau, t) = \frac{e^2}{c} (\beta_z)^{1/2} \delta(t - \tau \bmod T) \int_{-\infty}^t dt' w(t-t') \bar{D}(-t', t') \quad (4)$$

where  $w(t)$  is the transverse wake-potential associated with the structure and  $T$  the ring revolution period.

Multiplying the Fokker-Planck equation (2) by  $z$  and  $p_z$  respectively, using (3) and integrating by parts over  $z$  and  $p_z$ , we can derive two coupled partial differential equations for  $D(\tau, p_\tau, t)$  and  $P(\tau, p_\tau, t)$

$$\begin{aligned} (\delta/\delta t)D &= P, \\ (\delta/\delta t)P &= -\omega_z^2 D - 2\alpha_z P + (\beta_z)^{1/2} \omega_z \rho(\tau, p_\tau, t) F(\tau, t) / \gamma mc \end{aligned} \quad (5)$$

where  $\rho(\tau, p_\tau, t)$  is the longitudinal distribution function of the bunch and the operator  $(\delta/\delta t) = (\partial/\partial t) - L_s$  can be considered as a generalized total time derivative, taking into account dissipative and diffusive effects due to synchrotron radiation. Eqs.(5) do not contain the rms transverse beam radius  $\sigma_z$  and this shows that, in dipole approximation, the betatron motion is not affected by quantum excitation.

The longitudinal distribution satisfies the "reduced" Fokker-Planck equation  $(\delta/\delta t)p = 0$  and thus relaxes to a Gaussian steady state  $\rho(\tau, p_\tau)$ . We can combine the two coupled equations (5) to eliminate  $P$ , thus obtaining:

$$\begin{aligned} (\delta/\delta t)^2 D + 2\alpha_z (\delta/\delta t)D + \omega_z^2 D &= \\ &= (\beta_z)^{1/2} \omega_z \rho(\tau, p_\tau) F(\tau, t) / \gamma mc. \end{aligned} \quad (6)$$

This is formally identical to the equation of a damped harmonic oscillator driven by a force proportional to  $F(\tau, t)$ , but the further dependence on the synchrotron variables  $\tau$  and  $p_\tau$  has to be considered. To first order in the ratio  $\alpha_z/\omega_z$ , Eq.(6) has the following solution

$$D(\tau, p_\tau, t) = [(\beta_z)^{1/2} / \gamma mc] \int_{-\infty}^t dt' \exp[-\alpha_z(t-t')] \sin[\omega_z(t-t')] f(\tau, p_\tau, t, t'), \quad (7)$$

provided the function  $f(\tau, p_\tau, t, t')$  satisfies the reduced Fokker-Planck equation  $(\delta/\delta t)f = 0$  and takes the initial value  $f(\tau, p_\tau, t, t) = \rho(\tau, p_\tau) F(\tau, t)$  for  $t' = t$ . Indeed the r.h.s. of Eq.(7) describes the effect of the collective force experienced at all previous times  $t' \leq t$  by particles with final synchrotron coordinates  $\tau$  and  $p_\tau$ . Terms of second order in the ratio  $\alpha_z/\omega_z$ , which for LEP at injection energy is about  $10^{-6}$ , are negligible.

The function  $f(\tau, p_\tau, t, t')$ , appearing in Eq.(7), can be expressed through the Green's function of the reduced Fokker-Planck equation:

$$f(\tau, p_\tau, t, t') = \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} dp'_\tau G(\tau, p_\tau, \tau', p'_\tau, t-t') \rho(\tau', p'_\tau) F(\tau', t'). \quad (8)$$

The Green's function  $G(\tau, p_\tau, \tau', p'_\tau, t-t')$  represents the transition probability, in a time interval  $t - t'$ , from a point with synchrotron coordinates  $\tau'$  and  $p'_\tau$  to a point of coordinates  $\tau$  and  $p_\tau$ .

Since  $p_\tau$  is the only variable directly affected by quantum excitation, the Fokker-Planck operator  $L_s$  does not contain second order derivatives with respect to  $\tau$  (see Eq.(3)). This situation is similar to that encountered in the theory of Brownian motion and leads to a complicated formula for the transition probability [6]. Nevertheless, we can approximate  $G$  by

$$G(\tau, p_\tau, \tau', p'_\tau, t-t') = (1/2\pi\omega_s\sigma_s^2) \exp\{-[(p_\tau - \bar{p}_\tau)^2 + \omega_s^2(\tau - \bar{\tau})^2] / [2\omega_s^2\sigma_s^2(1-Q^2)]\}, \quad (9)$$

where  $\bar{\tau}(\tau', p'_\tau, t-t')$  and  $\bar{p}_\tau(\tau', p'_\tau, t-t')$  describe the deterministic damped oscillations of a particle with initial coordinates  $\tau'$  and  $p'_\tau$  at time  $t = t'$ , and  $Q(t-t') = \exp[-\alpha_s(t-t')]$ . The exact Green's function of the reduced Fokker-Planck equation differs from (9) by terms of order  $\alpha_s/\omega_s$ , which become important only for time intervals  $t - t'$  much shorter than the synchrotron period. Thus the approximate solution (9) can be used to investigate the effects of radiation damping and quantum excitation on the instability stop-bands at currents below threshold, characterized by rise times longer than the synchrotron period.

As in the previous paper [5], we focus our attention on the function  $\bar{D}_n(\tau) = \bar{D}(\tau, nT - \tau)$ , describing the transverse dipole density at the fixed azimuth  $\theta_0$  after  $n$  machine revolutions. Then, from Eqs.(4), (7), (8) and neglecting both synchrotron and betatron phase advances over a bunch length, we obtain the following integral equation

$$\begin{aligned} \bar{D}_n(\tau) &= (e^2\beta_z/E) \int_{-\infty}^{\infty} dp'_\tau \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} dp'_\tau \sum_{k=0}^{\infty} \exp(-k\alpha_z T) \sin(k\omega_z T) \\ &G(\tau, p_\tau, \tau', p'_\tau, kT) \rho(\tau', p'_\tau) \int_0^{\infty} dx w(x) \bar{D}_{n-k}(x + \tau'), \end{aligned} \quad (10)$$

where  $E = \gamma mc^2$  is the relativistic particle energy.

### 3. RESONANT COEFFICIENTS

By a Fourier analysis, the integral equation (10) can be reduced to the eigenvalue problem

$$D_q(v) = \sum_{p=-\infty}^{\infty} A_{qp}(v) D_p(v). \quad (11)$$

Here  $D_q(v)$  denotes the Fourier transform of the dipole density and the Fourier frequency  $\omega = (q+v)\omega_0$  has been split into an integer multiple  $q$  of the ring revolution frequency  $\omega_0$  plus a fractional tune  $v$ , whose real part is in the range  $[-1/2, 1/2]$ . The matrix  $A_{qp}(v)$  reads

$$A_{qp}(v) = K Z_p(v) \sum_{n=0}^{\infty} C_n(v) H_n[\sigma(q+v)] H_n[\sigma(p+v)], \quad (12)$$

where  $K = i e I_b \beta_z / 2E$  is a coefficient proportional to the bunch current and  $Z_p(v)$  is the transverse impedance of the localized structure, i.e. the Fourier transform of the wake-potential  $w(t)$ . The functions  $H_n[\sigma(q+v)]$  represent the so called Hermite modes [2] and depend on the dimensionless bunch length  $\sigma = \omega_0 \sigma_s$ .

The resonant coefficients  $C_n(\nu)$  are given by

$$C_n(\nu) = \sum_{m=0}^n \binom{n}{m} \frac{\sin\{2\pi[\nu_z - (n-2m)\nu_s]\}}{\cos\{2\pi[\nu - i(\Delta_z + n\Delta_s)]\} - \cos\{2\pi[\nu_z - (n-2m)\nu_s]\}} \quad (13)$$

where  $\Delta_z = \alpha_z/\omega_0$  and  $\Delta_s = \alpha_s/\omega_0$ . They carry all the information about the reflection properties of the synchro-betatron satellites  $\nu = \nu_z + m\nu_s$  and show that the imaginary frequency shift due to longitudinal damping is proportional to the mode number  $|m|$ .

Expanding the eigenvectors  $D_n(\nu)$  of Eq.(11) in the Hermite basis  $H_n[\sigma(q+\nu)]$ , we obtain a dispersion relation giving the fractional tune  $\nu$  of the orthogonal dipole modes as a function of the bunch current  $I_b$

$$\det \{ \delta_{nm} - K C_n(\nu) M_{nm}(\nu) \} = 0. \quad (14)$$

The impedance matrix  $M_{nm}(\nu)$  can be computed using a broad band resonator model as in [5].

4. NUMERICAL RESULTS

We have plotted the imaginary part of  $\nu$  versus  $I_b$  for three different values of the betatron tune  $\nu_z$ , chosen so that mode coupling occurs approximately at the same current below threshold. These figures have been obtained by a numerical solution of Eq.(14), with  $\nu_s = 0.088$  and with the other parameters corresponding to LEP at injection energy [5]. In particular, the dimensionless damping constants are  $\Delta_z = 3.5 \cdot 10^{-5}$  and  $\Delta_s = 7.0 \cdot 10^{-9}$ . Above threshold, the instability is always due to the coupling between modes 0 and -1.

Fig.1 shows an instability stop-band associated with the coupling between modes -3 and -1: we can identify the unstable mode as mode -1, since for vanishing current the imaginary part of its coherent tune approaches the value  $\Delta_z + \Delta_s$ . Actually this asymptotic value is slightly different, because even for vanishing current the Hermite modes are not exact eigenvectors of Eq.(11). Fig.2 shows a stop-band due to the coupling between modes -4 and 0 and, in this case, it is mode 0 which becomes unstable. Finally, in Fig.3 we consider the coupling between modes 5 and -1: here the instability stop-band disappears, since the imaginary part of  $\nu$  stays positive up to the threshold current  $I_{th}$ . We should mention that, for a given pair of coupling modes, the width of a stop-band and the maximum growth rate of the instability depend on the value of  $\nu_z$ . In particular, all the stop-bands occurring at very low current near the resonant values  $\nu_z = n \nu_s$  and  $\nu_z = 1/2 - n \nu_s$  disappear.

In conclusion, starting from the Fokker-Planck equation, we have computed the rise time of instabilities due to transverse mode coupling for localized impedances in presence of damping and noise. The approach is similar to that based on the Vlasov equation [5], but it shows the different role played by longitudinal and transverse damping. Betatron oscillations have the same imaginary frequency shift for all dipole modes, whereas the shift associated with synchrotron motion is proportional to the mode number. This gives rise to a splitting of the imaginary parts of the coherent tune  $\nu$ , thus allowing to identify the unstable modes. For high order dipole modes, damping is so strong that the instability stop-bands disappear completely.

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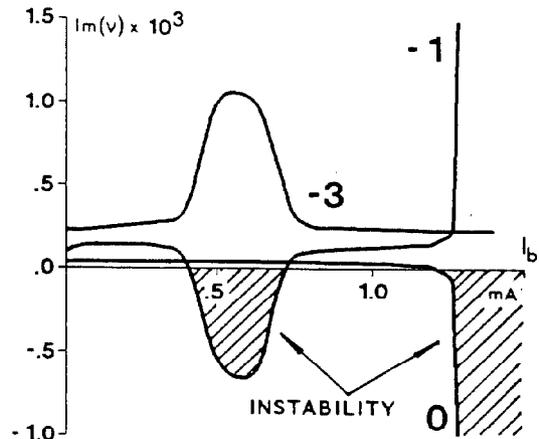


Fig.1 Coupling between modes -3 and -1:  $\nu_z = 0.174$ .

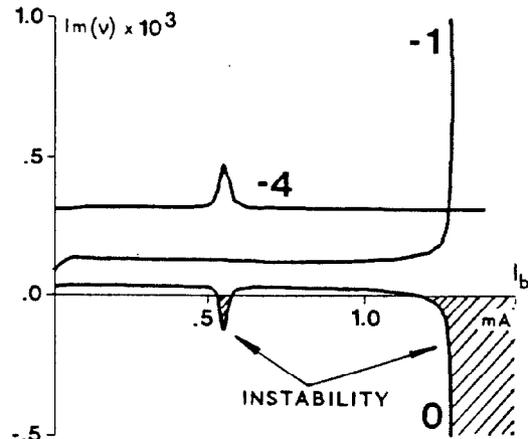


Fig.2 Coupling between modes -4 and 0:  $\nu_z = 0.185$ .

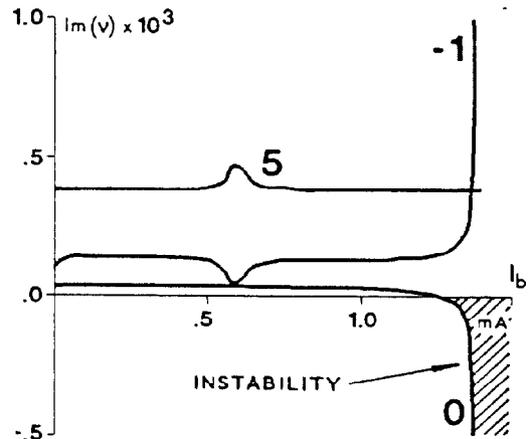


Fig.3 Coupling between modes 5 and -1:  $\nu_z = 0.322$ .