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SATURATION OF A LONGITUDINAL INSTABILITY DUE TO NONLINEARITY OF THE WAKE FIELD*

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Abstract

Self-sustained synchrotron oscillations are observed in electron storage rings. In general the theoretical description of the saturation of an instability for large oscillation amplitude is a difficult problem, and techniques have not yet been developed which yield analytic approximations to the appropriate nonlinear Vlasov or Fokker-Planck equations. In this paper, a single point bunch interacting with the wake field from a single resonant mode of an RF cavity is considered, and the averaging method of Bogoliubov and Mitropolsky[1] is used to study the saturation of the initial exponential growth of the oscillation amplitude, due to the nonlinearity of the wake field. The determination of the limiting amplitude of oscillation is discussed both in the presence and in the absence of radiation damping.

Equations of Motion

In a storage ring, the electrons in a bunch execute synchrotron oscillations with angular frequency ω_s about a synchronous electron having energy E_o and revolution period T_o. Suppose there is only a single bunch of N electrons in the ring, and that this bunch is interacting with the wake field due to a resonant mode of an RF cavity. The resonant frequency of the mode is ω_r, its shunt impedance is R and the quality factor is Q. The wake field w(t) vanishes for $t < 0$, and for $t > 0$ is given by

$$w(t) = 2\Gamma R \exp(-\Gamma t)\left[\cos\omega_r t - \frac{1}{2Q}\sin\omega_r t\right] \tag{1}$$

where $\Gamma = \omega_r/2Q$. We shall assume that Q is large enough to allow neglect of the $\sin\omega_r t$ term in Eq. (1). At time $t = 0$, the wake field $w(0) = \Gamma R$.

Let us assume the bunch length is short compared to the wavelength of the resonant mode so the consideration of a point bunch is justified. We denote the time displacement of the point bunch from the synchronous particle by $\tau(t)$, which is taken to be positive when the bunch leads the synchronous particle. Assuming the change in one revolution of the synchrotron oscillation phase and amplitude is small, the difference equations describing the interaction of the bunch with a discrete cavity located at a particular azimuth in the ring can be replaced by the differential equation of motion:

$$\ddot{\tau}(t) + \omega_s^2\tau(t) = \frac{\alpha_c Ne^2}{E_o T_o}\sum_{p=o}^{\infty} w\left(pT_o + \tau(t-pT_o)-\tau(t)\right), \tag{2}$$

where α_c is the momentum compaction of the storage ring and e is the charge of the electron.

It is convenient to introduce the dimensionless variables:

$$\begin{aligned} s &= \omega_s t, \\ x(s) &= \omega_r\tau(s/\omega_s), \\ \alpha &= \omega_r T_o, \\ \mu &= \omega_s T_o, \\ \lambda &= \frac{\omega_r}{\omega_s^2}\,\frac{\alpha_c Ne^2}{E_o T_o}\,2\Gamma R. \end{aligned}$$

We consider $\mu = \omega_s T_o \ll 1$, $\alpha = \omega_r T_o \gg 1$, and ΓT_o on the order of unity. If $\Gamma\tau(t) \ll 1$ for all times, then Eq. (2) is well approximated by:

$$d^2x(s)/ds^2 + x(s) = \lambda\sum_{p=1}^{\infty}\exp\left(-p\Gamma T_o\right)\cos\left(p\alpha + x(s-p\mu) - x(s)\right), \tag{3}$$

where the $p = 0$ term has been dropped since it only corresponds to a change of the stable phase. Introducing action-angle variables J,θ by

$$\begin{aligned} x(s) &= \sqrt{J(s)}\,\cos\theta(s), \\ dx/ds &= -\sqrt{J(s)}\,\sin\theta(s), \end{aligned} \tag{4}$$

we make the approximation,

$$x(s - p\mu) - x(s) \approx \sqrt{J}\,(\cos(\theta - p\mu) - \cos\theta), \tag{5}$$

in Eq. (3), which will be valid for small enough p, if the wake field does not alter the synchrotron motion rapidly compared to the synchrotron frequency. Assuming that values of p for which Eq. (5) is violated make a negligible contribution to the sum on the right hand side of Eq. (3), the equations of motion for J and θ are found to be:

$$dJ/ds = -2\lambda\sqrt{J}\sin\theta\sum_{p=1}^{\infty}\exp(-p\Gamma T_o)\cos\left(p\alpha + 2\sqrt{J}\sin\frac{p\mu}{2}\,\sin(\theta - \frac{p\mu}{2})\right)$$

$$d\theta/ds = 1 - \frac{\lambda}{\sqrt{J}}\cos\theta\sum_{p=1}^{\infty}\exp(-p\Gamma T_o)\cos\left(p\alpha + 2\sqrt{J}\sin\frac{p\mu}{2}\sin(\theta - \frac{p\mu}{2})\right). \tag{6}$$

The lowest-order averaging approximation is obtained by averaging the right-hand side of Eqs. (6) over $0 \le \theta \le 2\pi$. This will be valid when λ is small enough to assure that the wake field produces significant change in the synchrotron motion only on a time scale long compared to the synchrotron oscillation period. Upon averaging the right-hand side of Eqs. (6) over θ, we obtain

$$dJ/ds = 2\lambda\sqrt{J}\sum_{p=1}^{\infty}\exp(-p\Gamma T_o)\cos\frac{p\mu}{2}\sin p\alpha\, J_1\left(2\sqrt{J}\sin\frac{p\mu}{2}\right),$$

$$d\theta/ds = 1 - \frac{\lambda}{\sqrt{J}}\sum_{p=1}^{\infty}\exp(-p\Gamma T_o)\sin\frac{p\mu}{2}\sin p\alpha\, J_1\left(2\sqrt{J}\sin\frac{p\mu}{2}\right), \tag{7}$$

where $J_1(z)$ is the Bessel function of order unity.

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In the limit of small oscillation amplitude, $\sqrt{J} \ll 1$, Eqs. (7) can be linearized and we obtain

$$dJ/ds = \lambda J \sum_{p=1}^{\infty} \exp(-p\Gamma T_o)\sin p\mu \sin p\alpha \ ,$$
$$d\theta/ds = 1 - \frac{\lambda}{2}\sum_{p=1}^{\infty} \exp(-p\Gamma T_o)(1 - \cos p\mu)\sin p\alpha \ , \tag{8}$$

which corresponds to an exponential growth of the action, $J = \exp(2s/\omega_s\tau_{gr})$ and a shift in the oscillation frequency $d\theta/ds = 1 - \delta\omega/\omega_s$. Defining the complex coherent frequency shift

$$\Omega = \delta\omega + i/\tau_{gr} \tag{9}$$

we see that Eqs. (8) imply

$$\frac{\Omega}{\omega_s} = \frac{\lambda}{2}\sum_{p=1}^{\infty} \exp(-p\Gamma T_o)\sin p\alpha\left(\exp(ip\mu)-1\right) \ . \tag{10}$$

Writing this in terms of the impedance $Z(\omega) = Z^*(-\omega)$, which is the Fourier transform of the wake field, one finds

$$\Omega = \frac{ie\alpha_c I_{av}\omega_o}{4\pi E_o\omega_s}\sum_{n=-\infty}^{\infty}\left[(n\omega_o+\omega_s)Z(n\omega_o+\omega_s)-n\omega_o Z(n\omega_o)\right] \ , \tag{11}$$

where $\omega_o = 2\pi/T_o$ is the revolution frequency of the bunch. Eqs. (10) and (11) are the well-known results[2] valid in the regime of exponential growth.

The saturation of the exponential growth occurs because the Bessel functions appearing in Eqs. (7) are linear only for small values of their arguments. As the argument of the Bessel function increases, $J_1(z)$ increases more slowly than linear, and it eventually turns over and decreases to zero and then negative values. The limiting value of the action J is determined by the vanishing of the sum on the right-hand side of the first of Eqs. (7), assuring $dJ/ds = 0$.

A Simplified Model

In order to simplify the algebra in the following discussion, we shall keep only the p = 1 term in Eqs. (6) and consider the factor $\exp(-\Gamma T_o)$ to be absorbed into the parameter λ. Then Eqs. (6) reduce to

$$dJ/ds = -2\lambda\sqrt{J}\sin\theta\cos\left(\alpha + 2\sqrt{J}\sin\frac{\mu}{2}\sin(\theta - \frac{\mu}{2})\right)$$
$$d\theta/ds = 1 - \frac{\lambda}{\sqrt{J}}\cos\theta\cos\left(\alpha + 2\sqrt{J}\sin\frac{\mu}{2}\sin(\theta - \frac{\mu}{2})\right) \ . \tag{12}$$

Bogoliubov and Mitropolsky[1] have shown how the averaging method can be formulated to obtain a systematic asymptotic expansion in powers of λ. To proceed, one considers the right-hand side of Eqs. (12) to be expanded in Fourier series,

$$dJ/ds = \lambda\sum_{n=-\infty}^{\infty} G_n(J)\exp(in\theta) \ ,$$
$$d\theta/ds = 1 + \lambda\sum_{n=-\infty}^{\infty} A_n(J)\exp(in\theta) \ , \tag{13}$$

and transformed action-angle variables I and ψ are introduced via,

$$J = I + \lambda\xi(I,\psi) + O(\lambda^2),$$
$$\theta = \psi + \lambda U(I,\psi) + O(\lambda^2) \ . \tag{14}$$

Then ξ and U are determined from the condition that the equations of motion for I and ψ have the form

$$dI/ds = \lambda X_o(I) + \lambda^2 X_1(I) + O(\lambda^3) \ ,$$
$$d\psi/ds = 1 + \lambda\Omega_o(I) + \lambda^2\Omega_1(I) + O(\lambda^3) \ , \tag{15}$$

where the right-hand side of Eqs. (15) are independent of ψ. One finds that

$$\xi = \sum_{n\neq o}\frac{G_n(I)}{in}\exp(in\psi) \ ,$$
$$U = \sum_{n\neq o}\frac{A_n(I)}{in}\exp(in\psi) \ , \tag{16}$$

and

$$dI/ds = \lambda G_o(I) - \lambda^2\sum_{n\neq o}\left[\frac{dG_n(I)}{dI}G_{-n}(I)\frac{1}{in} + G_n(I)A_{-n}(I)\right] \ , \tag{17a}$$

$$d\psi/ds = 1 + \lambda A_o(I) - \lambda^2\sum_{n\neq o}\left[\frac{dA_n(I)}{dI}G_{-n}(I)\frac{1}{in} + A_n(I)A_{-n}(I)\right] \ . \tag{17b}$$

The limiting amplitude of oscillation can be determined from Eq. (17a) without reference to Eq. (17b). Computing the Fourier coefficients $G_n(I)$ and $A_n(I)$, we can rewrite Eq. (17a) as

$$dI/ds = 2\lambda\sqrt{I}\sin\alpha\cos\frac{\mu}{2}J_1(z) + \lambda^2\sin^2\mu\, f(z,\alpha) \ , \tag{18}$$

where

$$f(z,\alpha) = 2\sin^2\alpha\sum_{k=o}^{\infty}J^2_{2k+1}(z) + 2\cos^2\alpha\sum_{k=1}^{\infty}J^2_{2k}(z) \ , \tag{19}$$

and

$$z = 2\sqrt{I}\sin(\mu/2) \ . \tag{20}$$

Note that $f(z,\alpha)$ is always positive.

Let the radiation damping time for synchrotron oscillations be τ_{rad}, and for now suppose λ is small enough to allow neglect of the second-order term, $\lambda^2\sin^2\mu\, f(z,\alpha)$, on the right-hand side of Eq. (18). It then follows that

$$\frac{1}{I}\frac{dI}{dt} = \frac{4}{\tau_{gr}}\frac{J_1(\mu\sqrt{I})}{\mu\sqrt{I}} - \frac{2}{\tau_{rad}} \ , \tag{21}$$

where $1/\tau_{gr}$ is the initial exponential growth rate in the absence of radiation damping,

$$\frac{1}{\tau_{gr}} = \frac{\lambda\omega_s}{2}\mu\sin\alpha \ , \tag{22}$$

and we have made the small angle approximation $\sin\mu \approx \mu$. Of course, $\sin\alpha$ is taken to be positive so that one has exponential growth, not decay.

When $\tau_{gr} < \tau_{rad}$ the oscillation amplitude will increase until it reaches the limiting value $\sqrt{I_o}$ determined by

$$\frac{J_1(\mu\sqrt{I_o})}{\mu\sqrt{I_o}} = \frac{\tau_{gr}}{2\tau_{rad}} \quad . \tag{23}$$

When τ_{gr} is not too much less than τ_{rad}

$$\mu\sqrt{I_o} \approx \sqrt{8\left(1 - \tau_{gr}/\tau_{rad}\right)} \ , \tag{24}$$

and when $\tau_{gr} \ll \tau_{rad}$,

$$\mu\sqrt{I_o} \approx z_1 \ , \tag{25}$$

where $z_1 \approx 3.83$ is the first zero of $J_1(z)$.

In the case, $\tau_{gr} \ll \tau_{rad}$, when the radiation damping is negligible, it is of interest to keep the second-order term on the right-hand side of Eq. (18), so that we can determine the dependence of the limiting oscillation amplitude $\sqrt{I_L}$ on the initial exponential growth rate $1/\tau_{gr}$. One finds

$$\sqrt{I_L} = \sqrt{I_o} + \frac{T_o}{\tau_{gr}} \frac{f(z_1,\alpha)}{z_1\left|J_o(z_1)\right|\sin^2\alpha} \tag{26}$$

where $\sqrt{I_o} = z_1/\mu$.

Concluding Remarks

Within the lowest-order averaging approximation the self-sustained synchrotron oscillation has the form $\tau(t) = A\sin\omega_s t$, where the amplitude A is determined by the considerations of the previous section. The signal induced on a pick-up electrode located at a fixed azimuth of the storage ring is proportional to the bunch density

$$\rho(t) = \sum_n \delta\left(t - nT_o - A\sin\omega_s t\right)$$

$$= \frac{1}{T_o}\sum_n e^{in\omega_o t}\sum_k e^{-ik\omega_s t} J_k\left(n\omega_o A\right) \ , \tag{27}$$

where $\omega_o = 2\pi/T_o$ is the angular revolution frequency. Although the synchrotron oscillation is sinusoidal with frequency ω_s, the higer-order sidebands at $k\omega_s$ $(k>1)$ will be observed on the pick-up for large amplitude A.

We have made a numerical check of the averaging approximation by considering the recursion relations:

$$\begin{aligned} p_{n+1} &= p_n - Kq_n + \lambda\sin p_n \\ q_{n+1} &= q_n + p_{n+1} \quad . \end{aligned} \tag{28}$$

Action-angle variables J_n, θ_n are introduced via

$$a_n \equiv \sqrt{J_n}e^{i\theta_n} = \left(e^{i\mu} - 1\right)q_n + p_n \quad , \tag{29}$$

where $K = 2(1 - \cos\mu)$. In terms of a_n, the recursion relations of Eq. (28) become

$$a_{n+1} = e^{i\mu}\left(a_n + \lambda\sin p_n\right) \quad . \tag{30}$$

For $\lambda = 0$, the action variable $J_n = Kq_n^2 + p_n\left(p_n - Kq_n\right)$ is a constant of the motion, but for $\lambda > o$ it increases with the initially exponential growth rate $J_n = \exp(\lambda n)$. We have found numerically that in the case of $K = 0.01$ and $\lambda = 0.002$, the value of $\sqrt{J_n}$ for $n > 20{,}000$ fluctuates with a variation of less than 0.5% about

$$\sqrt{J_n} = 3.83 \ . \tag{31}$$

Initial conditions of $q_n = 0.1$ and $p_n = 0$ were taken.

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References

1. N. N. Bogoliubov and Y. A. Mitropolsky, Asymptotic Methods in the Theory of Non-Linear Oscillations, p. 412 (Hindustan Publishing Corporation, India, 1961).

2. See e.g. A. W. Chao, Physics of High Energy Particle Accelerators, p. 353, M. Month, Ed., AIP Proc. No. 105 (1983).