COHERENT OSCILLATIONS PRODUCED BY THE RAPID SKIRTING OF AN INTEGRAL RESONANCE*

M.M. Gordon and Felix Marti

National Superconducting Cyclotron Laboratory, Michigan State University, East Lansing, MI 48824

Abstract

Although it is well known that accelerating a cyclotron beam through an integral resonance generates a coherent oscillation of the beam, it is much less widely known that such oscillations can also be produced just by rapidly skirting the same resonance. The latter will occur whenever the perturbing field or the radial (vertical) tune changes so rapidly in the vicinity of the resonance that the accelerating beam cannot smoothly follow the resultant non-adiabatic shift of the equilibrium orbit. A straightforward analysis of these effects leads to a simple formula for the resultant growth of the oscillation amplitude. This formula is applied to an example recently found in superconducting cyclotrons when the radial tune dips rapidily, but briefly, down close to unity. Applications to other cyclotrons are also considered.

1. INTRODUCTION

Most nonrelativistic cyclotrons make use of the $\nu_r = 1$ resonance to generate a coherent oscillation of the beam. Acceleration through this resonance is frequently used in the extraction process where the resultant coherent oscillation helps the beam to clear the septum and enter the electrostatic deflector.

In the central region, this resonance phenomenon is often used to adjust the centering of the beam. In certain cases, however, the coherent oscillation is produced by a rapid skirting of the resonance rather than an acceleration through it, although the difference is not always clear.

In the central region of the Triumf cyclotron, for example, an isochronous field is used rather than a magnet cone, and the resultant value of (ν_r-1) remains above zero even at the injection energy.[1] But in this case, the acceleration process complicates the situation during the early turns since, in addition to producing needed vertical focusing, it also tends to drive the radial oscillations into a stopband where (ν_r-1) takes on imaginary values.[2]

A clear-cut example of the resonance skirting phenomenon has recently been found in superconducting cyclotrons when there is a large difference in current in two adjacent trim coils. The resultant change in field gradient produces a fairly sharp dip in the ν_r vs. E curve which temporarily brings the value down close to unity.

This type of behavior is evident, for example, in the ν_r curve shown in Fig. 1 which was obtained in the case of C^{4+} ions having a final energy of 30 MeV/A. Here the value of ν_r drops from 1.065 at 20 MeV/A down to 1.027 at 22 MeV/A and then rises to 1.082 at 24 MeV/A.

The resonance skirting phenomenon can be readily demonstrated by examining the results from computed orbits. Fig. 2 shows, for example, a plot of p_x vs. x for one such orbit between 19 and 25 MeV/A where the energy gain per turn is 70 keV/A. The (x,p_x) points here give the deviation of (r,p_r) from the equilibrium orbit (EO) value at each energy, and we should note that all orbits are calculated in a realistic magnetic field which includes, in particular, a first harmonic (n=1) component having an amplitude b_1 = 4.3 Gauss in this region. It is this component, of course, which drives the resonance and which is also responsible for the effects considered here.

The solid curve in Fig. 2 represents (x,p_x) points plotted once per turn with every tenth turn marked by a cross. For the first 14 turns between 19 and 20 MeV/A, the points execute a very small loop, which indicates that the orbit is very well centered on the "accelerated" EO. For the next 27 turns, the points trace out a second loop and arrive at 22 MeV/A where the ν_r curve in Fig. 1 shows a pronounced minimum. At this energy, the curve in Fig. 2 exhibits a cusp, which indicates an abrupt change in its development.

From here on, the curve swings rapidly outward and ends on a much larger loop having a radius of about 0.8 mm. Thus, if this orbit represented the central ray of a cyclotron beam, one would observe that the beam developed a coherent oscillation with an amplitude of about 0.8 mm between 20 and 24 MeV/A. Clearly, this effect results from the resonance skirting described above.

A fairly simple theory has been developed which allows one to calculate the coherent amplitude produced either by traversing or by rapidly skirting any integral resonance. The results can be put into a form which explicitly distinguishes between an adiabatic and a non-adiabatic process.

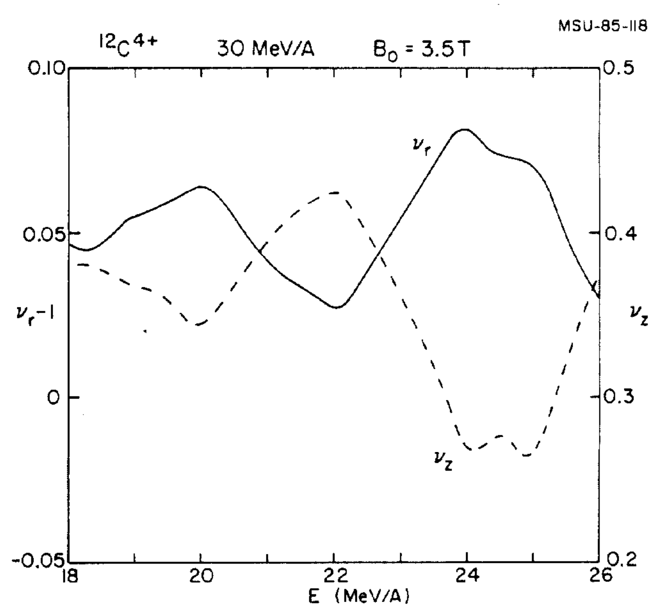

Fig. 1 Plot of ν_r (left scale) and ν_z (right scale) vs. E for a magnetic field designed to accelerate C^{4+} ions to 30 MeV/A. The ν_r curve exhibits a pronounced minimum near 22 MeV/A, and this rapid change in ν_r leads to the resonance skirting effect.

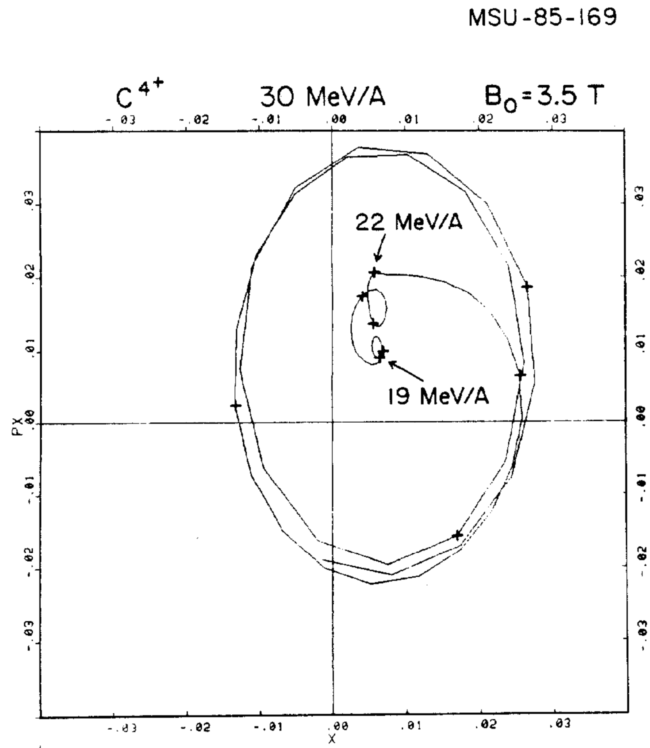

Fig. 2 The solid curve represents (x,p_x) points plotted once per turn for an orbit accelerated from 19 to 25 MeV/A with an energy-gain per turn qV = 70 keV/A. Crosses mark the points on every tenth turn. The orbit depicted here starts out very well centered and ends up with an oscillation amplitude ΔA = 0.03 inch as a result of skirting the $\nu_r = 1$ resonance. Note that p_x here is actually $p_x/m\omega$.

2. BASIC THEORY AND RESONANCE TRAVERSAL

The theory used here is simply an extension of that used in a previous paper concerning the perturbations produced by gap crossings.[3] Here again we start from the smooth approximation equations of motion:

$$dp_x/d\Theta = -\nu^2(xp/R) + f(\Theta), \qquad (1a)$$

$$dx/d\Theta = (Rp_x/p) + g(\Theta), \qquad (1b)$$

where $\nu = \nu_r$ here, and where $p/R = qB_0(R)$ is almost constant.

The terms $f(\Theta)$ and $g(\Theta)$ represent the perturbations produced by field errors and gap crossings. The latter were discussed at length in a previous paper,[3] and we now consider in more detail the effect of a field error ΔB_z given by

$$\Delta B_z = \sum_n b_n(R)\cos(n\Theta-\delta_n(R)), \qquad (2)$$

which is evaluated at the average radius R. In this case, $f(\Theta) = qR\Delta B_z$.

We next introduce the rotating vector

$$X = x + i\beta p_x, \qquad (3)$$

where $\beta = R/p\nu$. When there are no perturbations, the solution can be expressed in terms of standard action and angle variables, J and ψ, as follows,

$$X(\Theta) = (2J\beta)^{1/2}\, e^{-i\psi}, \qquad (4a)$$

$$\text{where}\quad \psi = \int \nu\, d\theta. \qquad (4b)$$

Thus for example, the (x,p_x) points plotted in Fig. 2 rotate clockwise by an angle $2\pi(\nu-1)$ per turn in the absence of perturbations. The action J is, of course, an adiabatic invariant.

The differential equation for X follows directly from (1), namely,

$$(dX/d\Theta) + i\nu X = G(\Theta) + \varepsilon(X-X^*), \qquad (5)$$

$$\text{where}\quad G(\Theta) = g(\Theta) + i\beta f(\Theta), \qquad (6)$$

and where $\varepsilon = \beta'/2\beta$, with $\beta' = d\beta/d\Theta$.

The extra ε term leads mainly to the adiabatic damping factor $\sqrt{\beta}$ which occurs in X of (4a), and since this factor changes so little over the energy range of interest here, we shall henceforth drop this ε term.

Under these conditions, the complex oscillation amplitude $A(\Theta)$ remains constant when G = 0, where

$$A(\Theta) = X(\Theta)\, e^{i\psi}, \qquad (7)$$

with ψ given in (4b). In terms of this amplitude, integration of (5) then yields

$$A(\Theta) = A_i + \int G(\Theta)\, e^{i\psi}\, d\Theta, \qquad (8)$$

where A_i is the initial amplitude, and where the integration runs from Θ_i to Θ. This simple formula shows directly how the perturbation affects the amplitude.

The perturbation G is next written as a fourier series:

$$G(\Theta) = \sum_k G_k(R)\, e^{-ik\Theta}, \qquad (9)$$

where the sum extends over k = 0, ±1, ±2, etc. Inserting this series into (8) above, we find

$$A(\Theta) = A_i + \sum_k \int G_k(R)\, e^{i(\psi - k\Theta)}\, d\Theta, \qquad (10)$$

and we note that in the vicinity of ν=n, the term with k=n predominates. In particular, for the field error given in (2), we obtain

$$G_n(R) = i(Rb_n/2\nu B_0)\exp(i\delta_n). \qquad (11)$$

To obtain the complete G_n, the contribution of the gap crossings should be added on.[3]

Because of the acceleration, R and hence G_n depends indirectly on Θ, but this dependence is generally weak except near the center of the cyclotron. In this connection, we note that when R is smaller than the magnet gap, the amplitude b_n falls off to zero in proportion to R^n.

Thus, even though ν=1 at R=0 for an isochronous field, the perturbation is ineffective there since G_1=0 at R=0. In this case, the resonance can only be skirted and even then, the effect will occur at a radius comparable to the magnet gap. These conclusions are consistent with the calculations and measurements carried out for the Triumf cyclotron,[1] but as noted above, the picture here is somewhat clouded by the gap crossing effects.

We first apply the formula (10) to the old problem of resonance traversal. For this purpose, we assume that close to the resonance ν=n, the value of ν can be approximated by

$$\nu = n + \nu'(E-E_r), \qquad (12)$$

where $\nu' = d\nu/dE$ is evaluated at E_r, the resonance energy.

We now single out the term in (10) with k=n and consider its effect alone. In this case, the integral reduces to a complex Guassian or two real Fresnel integrals. Using G_n from (11), the resultant increase in oscillation amplitude is then found to be

$$|\Delta A| = (\pi R b_n/nB_0)(qV|\nu'|)^{-\frac{1}{2}}, \qquad (13)$$

where qV is the energy-gain per turn, and where all quantities are evaluated at the resonance.

This result agrees in all respects with that given by many other authors except that some of them manage to obtain an extra factor of $\sqrt{2}$. Note that in the central region, the gap crossing perturbations cited above are often important and should then be included. Note also that if the resonance occurs in the vertical motion with $\nu_z = n$, then the final result is the same except that b_n becomes the amplitude of zhe n^{th} harmonic of ΔB_r for z=0.

3. DISPLACED EO AND RESONANCE SKIRTING

The formula (10) for A(Θ) does not exhibit the EO displacement produced by the perturbations. That is, the important amplitude is the one characterizing the oscillations about the displaced (or accelerated) EO.

To obtain this amplitude, we carry out a partial integration of (10) and thereby obtain

$$A(\Theta) = a_i + A_{eo} - \sum_k \int (dC_k/d\Theta) e^{i(\psi - k\Theta)} d\Theta, \qquad (14)$$

where $\quad C_k = -iG_k/(\nu-k), \qquad (15)$

and, $\quad A_{eo} = (\sum_k C_k e^{-ik\Theta}) e^{i\psi}. \qquad (16a)$

That is, the coordinates of the displaced EO are given by the vector

$$X_{eo}(\Theta) = A_{eo} e^{-i\psi} = \sum_k C_k e^{-ik\Theta}, \qquad (16b)$$

in accordance with (7).

This formula for A(Θ) now exhibits the contribution from the displaced EO which includes, in general, the effect of the gap crossings. Note that the partial integration used above requires that ν≠n, and must therefore be modified somewhat if the resonance ν=n is actually traversed. This can be accomplished simply by isolating the term with k=n and treating it separately as in the preceding section.

The modified formula (14) above serves two purposes. First, it allows one to calculate directly the reduced amplitude a(Θ) defined by

$$a(\Theta) = (X(\Theta) - X_{eo}(\Theta)) e^{i\psi} = A(\Theta) - A_{eo}, \qquad (17)$$

which fulfills our objective. Thus, if E is constant, then $dC_k/d\Theta = 0$, and $a(\Theta) = a_i$, just as expected.

Second, (14) shows the distinction between an adiabatic and a non-adiabatic process. In the former case, every term in the sum is relatively small and oscillates rapidly, so that a(Θ) remains constant on the average. Thus, under adiabatic conditions, the accelerating ions will (except for small fluctuations) smoothly follow the changing displacement of the EO.

By contrast, a non-adiabatic process is characterized by a persistent increase in a(Θ) arising from the integral in (14) with k=n. To see this more clearly, we examine this term separately. From (15) and (11), we have

$$C_n(R) = (Rb_n/2\nu(\nu-n)B_0)\exp(i\delta_n), \qquad (18)$$

which shows that $dC_n/d\Theta$ will be exceptionally large under the conditions described in the abstract for rapidly skirting the ν=n resonance.

To obtain a formula for the growth in amplitude in a convenient form, we replace Θ by E in the integration by setting dΘ = λdE, where λ = 2π/qV. The resultant growth in amplitude obtained from (14) then becomes

$$|\Delta a(\theta)| = |\int (dC_n/dE) e^{i\phi}\, dE|, \qquad (19a)$$

where $\quad \phi(E) = \lambda \int (\nu - n) dE. \qquad (19b)$

This shows directly that as qV increases, the process becomes less adiabatic. Conversely, when qV is very small, λ and hence φ becomes very large, so that the integral becomes negligibly small, as expected.

Fig. 3 shows a plot of |Δa| as a function of qV, the energy gain per turn. The solid curve was calculated from the theoretical formula (19), and the main contribution in the factor (dC_1/dE) in this case comes from the variation of ν_r with E shown in Fig. 1. The plotted points in Fig. 3 were extracted from computer data like that given in Fig. 2. As can be seen, the agreement between the theory and the data is fairly good considering the simplicity of the theory.

We should note in conclusion that this phenomenon could influence the choice of first harmonic field bump used in the extraction process. That is, the Δa found here must be added vectorially to the amplitude growth subsequently produced when the beam traverses the $\nu_r = 1$ resonance in order to obtain the total amplitude of the coherent oscillations.

---

* This work supported by the National Science Foundation under Grant No. PHY-83-12245.

MSU-85-138

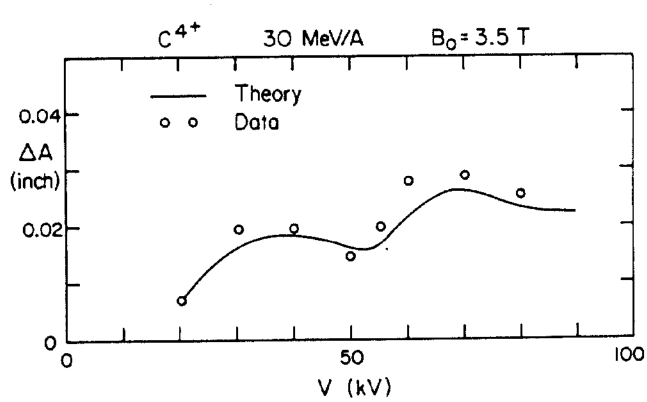

Fig. 3 Comparison of theory (solid curve) and computed orbit data (circled points) for the amplitude growth ΔA as a function of the dee voltage V, in keV/A. As V increases, the resonance is skirted more rapidly, and the process therefore becomes less adiabatic.

REFERENCES

1. J.L. Bolduc and G.H. MacKenzie, IEEE NS-18,287(1971);Cyclotrons-72(Am. Inst. Phys., New York,1972) 351.
2. M.M. Gordon and Felix Marti, Particle Accelerators12,13(1982).
3. M.M. Gordon, IEEE NS-30,2439(1983).