EXACT CALCULATION OF NONLINEAR ORBIT PROPERTIES OF A SYNCHROTRON

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Abstract

The trajectory of a particle in an arbitrary guide field can be computed as accurately as desired by numerical integration. This "measured" trajectory can then be fit with the phase-amplitude relation describing betatron oscillations, suitably generalized to account for nonlinear effects. Information about closed on- and off-momentum orbits, as well as the frequencies of small oscillations about them, can be obtained from such fits. Thus one can compute chromaticities exactly, among other things.

1. Introduction

Consider the very small alternating-gradient synchrotron shown in Fig. 1. Each 90° bend consists of six wedges having plane pole faces; thus the field in a quadrant is piecewise Cartesian rather than truly radial. When we add to this the additional complications of transition fields between the F and D wedges and fringe fields at the ends of each quadrant, we have a machine which is nonlinear even for small oscillations of a test particle. Similar effects are found in large machines, whose arcs are usually approximated by many relatively short straight magnets. In a small machine, where the transverse oscillation amplitude is more nearly comparable to the bend radius, such effects are more important.

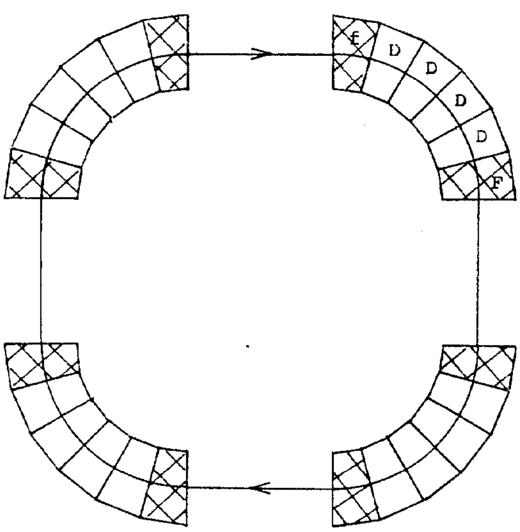

Fig. 1: An inherently nonlinear guide field. Wedges represent gradient magnets with plane pole faces. The field is piecewise Cartesian rather than radial.

One might wish to calculate, among other things, the distortion of the closed on-momentum orbit due to the non-radial character of the field; the shape of the closed off-momentum orbits and the chromaticity (frequencies of small oscillations about closed off-momentum orbits). Recent literature contains at least three distinct ways of attacking these problems, notably the chromaticity. Two, the methods of Jäger and Möhl[1] and that of Peggs[2], are second-order methods; the method of Dragt[3] is exact.

The method of Jäger and Möhl is basically an extension of the Courant tuneshift formula[4] to second order; they express the chromaticities in terms of integrals around the ring of the quadrupole and sextupole coefficients of the fields suitably weighted by the unperturbed curvature, beta and dispersion functions. The chromaticity can thus be written as the sum of contributions from each magnet, and its physical sources become clear. Their formulas are incorporated in the synchrotron modeling program COMFORT[5].

Peggs' method consists of finding a Twiss matrix for motion around the off-momentum orbit in terms of the first- and second-order transport matrices for the design orbit. The perturbed tunes are given by the trace of this matrix and can therefore be expressed directly in terms of the second-order matrix elements which in turn can be calculated by the program TRANSPORT. Jäger and Möhl found good agreement between their method and Peggs'.

Dragt's method starts with a complete specification of the magnetic field everywhere (rather than an optical equivalent or a finite multipole expansion). A numerical procedure is then used to simultaneously integrate the equations of motion for a particle trajectory and the variational equations for neighboring trajectories. A rapidly convergent Newton's search procedure is used to find closed orbits (which may be on- or off-momentum) and the solution to the variational equations provides the tunes of these orbits.

Functionally, the method we now describe resembles Dragt's: it starts with a complete field map and yields closed orbits and tunes. It too is exact, that is, limited by computational effort rather than any particular degree of approximation. Otherwise, it is completely different. Suppose that, starting with a complete field map, one has obtained by some means independent of synchrotron theory (e.g. numerical integration) a picture of the betatron oscillation of some test particle. An example is trace 1 of Fig. 2: a small oscillation about a distorted closed orbit highly displaced because of the 2% momentum defect. Even from a casual inspection of this picture we can say something about the tune: it is approximately 1. (The picture represents a full circuit of the machine.) If we had an equation whose parameters could be adjusted to fit the trace accurately, such a fit would give us accurate information about the tune. For instance, if we had picked conditions leading to something more nearly sinusoidal, we could certainly estimate the tune by fitting a sine function with arbitrary displacement, amplitude, frequency and phase.

Looking at Fig. 2 it is clear that, in the actual case, the fitting function must be much more complicated if we are to obtain a good fit. In fact, what it must be is precisely a generalization to the nonlinear case of the phase-amplitude relation familiar from beta theory! We have found such a trial function by guesswork, using the linear case as a model. To that extent our procedure is semi-empirical. Nevertheless the formula does yield excellent fits for small oscillations, and there is little reason to doubt the closed orbits and tunes we obtain, as we shall try to show.

2. Linear Case

In the linear approximation with median-plane symmetry the displacement x of the particle from the

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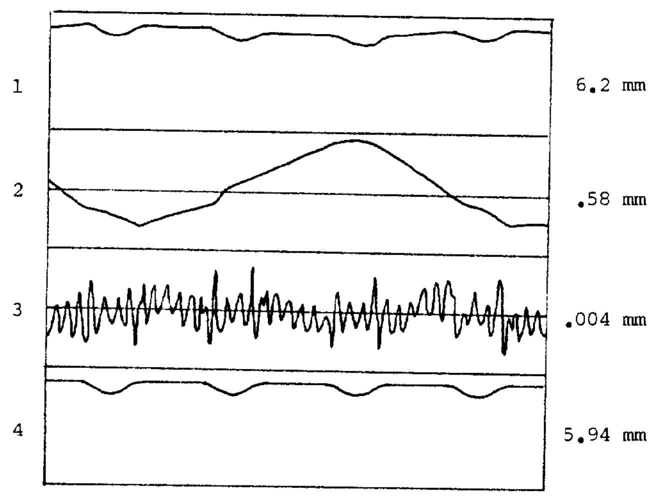

Fig. 2: Some results for a particle with momentum defect = .02 . 1) Small oscillations about displaced equilibrium orbit, output of numerical integration. 2) Error of linear equation (1). 3) Facsimile of error of fit equation (7). 4) h term of eqn. (7), analogous to dispersion function of linear theory. Numbers at right: peak value of each trace.

equilibrium orbit in the bend plane is given by

$$x(s) = A\, w(s) \cos(\Psi(s) + B) + \delta . \eta(s) \qquad (1)$$

s is the circumferential coordinate. A, B and δ are constants which depend on the initial conditions; in particular the momentum defect

$$\delta = (p - p_0) / p_0 \qquad (2)$$

is constant because we ignore acceleration. If the machine consists of N periods each of circumferential length L then the lattice functions w and η are periodic in s with period L; w is the square root of the familiar beta function while η is the dispersion function. Given the focusing function, w and η can be found by well-known methods. Ψ itself is not periodic; it is a monotonically increasing function of s with undulations, but its derivative is periodic and in fact, closed tied to the beta function:

$$\Psi' = \frac{d\Psi}{ds} = \frac{1}{\beta(s)} = \frac{1}{w^2} \qquad (3)$$

We can always choose the origin of s to be at an extremum of the lattice functions so that their derivatives vanish there; doing this, and also assuming the test particle starts there we can obtain simple expressions for A and B in terms of the initial conditions and lattice functions:

$$A = (x_0^2 + (x_0'/\Psi_0')^2)^{\frac{1}{2}} / w_0 \qquad (4)$$

$$B = - \tan^{-1} ((x_0'/\Psi_0')/ x_0) \qquad (5)$$

The tune Q of the orbit is the number of oscillations per circuit of the machine; if C = N L is the circumference of the ring then

$$2\pi Q = \Psi(s+C) - \Psi(s) = \Psi(C) \qquad (6)$$

Similar considerations apply to the vertical displacement except that the dispersion function vanishes. The gist of this review is that, if the starting position and slope of a particle are known, its trajectory is predicted by the linear theory. Little use is normally made of this fact, as we are not usually interested in the paths of individual particles.

3. Nonlinear Case

Here it is useful to think of three orbits. The layout orbit is simply a reference line around the machine; a convenient definition is the path resulting when all fringe fields are ignored and all multipoles past the dipole are turned off. It consists of line segments and arcs of circles, and no particle follows it exactly. The equilibrium orbit is the actual closed path of a non-oscillating particle, and depends on its momentum defect. Finally we have the actual orbits of particles in the machine, which consist of small or large betatron oscillations about equilibrium orbits. Let us assume that there exist periodic functions u, ϕ' and h of s such that the displacement x from the layout orbit is given by

$$x(\delta;s) = C\, u(\delta;s) \cos(\phi(\delta;s) + D) + h(\delta;s) \qquad (7)$$

There is a strong analogy to (1) but three important differences: a) The lattice functions depend implicitly on δ as indicated; b) We do not assume h(0;s) = 0; in fact h(0;s) is the equilibrium on-momentum orbit with respect to the layout orbit; c) No relation such as (3) is assumed *a priori* between ϕ' and u, since (3) was just a consequence of the linear approximation rather than the deeper assumption of the periodicity of the machine.

Now assume we have found by numerical integration an actual trajectory corresponding to some reasonable initial position, slope and momentum defect. Consider how we might fit this "measured" trajectory with (7) and what we might learn from such a fit. Clearly we would immediately have found the corresponding equilibrium orbit h(s); we will soon show that we will also know the tune, and we may be able to learn other things, such as a generalized beta function. To begin, reasoning as in the linear case we find

$$C = ((x_0 - h_0)^2 + (x_0'/\phi_0')^2)^{\frac{1}{2}} / u_0 \qquad (8)$$

$$D = - \tan^{-1} ((x_0'/\phi_0')/(x_0 - h_0)) \qquad (9)$$

But these equations contain the general lattice functions, which are not yet known. This involves us in a circular process. So let us rewrite (7)

$$x(s) = f(s) \cos\phi(s) + g(s) \sin\phi(s) + h(s) \qquad (10)$$

absorbing the constants C and D in the new periodic functions f and g. Let us impose periodicity on f, g, h and ϕ' by writing them as Fourier series e.g.

$$f(s) = f_0 + \sum_{j=1}^{m} f_j \cos(2\pi j\, s / L) \qquad (11)$$

with similar expressions for g and h and

$$\phi'(s) = p_0 + \sum_{j=1}^{m} p_j \cos(2\pi j\, s / L) \qquad (12)$$

Cosine series suffice because of our choice of origin for s. Now fit (10) to the measured trajectory; the forty or so Fourier coefficients are the adjustable parameters of the fit. (10) is linear in thirty of them, which simplifies matters somewhat. Now the coefficients are known. Integrating (12) gives

$$\phi(s) = p_0 s + \sum_{j=1}^{m} \frac{L}{2\pi j} p_j \sin(2\pi j\, s / L) \qquad (13)$$

and therefore

$$Q = \frac{\phi(C)}{2\pi} = \frac{N\,\phi(L)}{2\pi} = \frac{N\, L\, p_0}{2\pi} = p_0 R \qquad (14)$$

where R is the gross radius. We have found the tune in the nonlinear case.

One would like to go farther and find the nonlinear generalization u of the function w. However we find, upon doing the algebra, that the fit has only determined the product C.u of (7). In other words (7) alone does not unambiguously define an amplitude function; some subsidiary condition is needed. This question is left to ponder.

4. Computational Details and Results

We have written a program to implement these ideas. After accepting parameters defining a machine and a test particle, it performs a conventional 3 x 3 transport matrix analysis to determine the linear lattice functions; these will be used as a starting point for the nonlinear fit. By means of the Runge-Kutta algorithm the test particle is then tracked around the ring (we need to follow it for approximately one betatron period). The trajectory thus found is fit as follows: a starting set of ϕ' coefficients is found by means of a linear-least-squares fit to the ψ' function (3) found in the linear analysis. These are then improved, starting with p_0, by a step-search using as a figure-of-merit the logarithm of the fit residual normalized to the rms deviation of the trajectory from its mean. Near a good fit this quantity is parabolic in each p_j which is used to speed up the fit. After each change of a p_j the thirty remaining coefficients are recomputed by linear least-squares (matrix inversion). Only one pass through the p_j is needed. The entire process, in both x and y, takes about 7 minutes CPU time on a VAX 11/780; this is strongly dominated by the fit search. One can stop with p_0 and get the correct tune, but the fit will be poor.

We have obtained many results for a machine of the type shown in Fig. 1. Fig. 2 is typical: it shows 1) the trajectory of a slightly oscillating 2% off-momentum particle; 2) the fit error for the linear form (1) (about 9%); 3) the fit error for the nonlinear form (7) (about .05%) and 4) the closed orbit h(.02;s). The actual trajectory and the linear prediction would be easily distinguishable in a side-by-side comparison; the actual trajectory and the non-linear fit would be completely indistinguishable. The character of the fit error, rapidly oscillating about a fairly flat baseline, shows it is dominated by the finite number of Fourier coefficients used (10 here for each of the four functions).

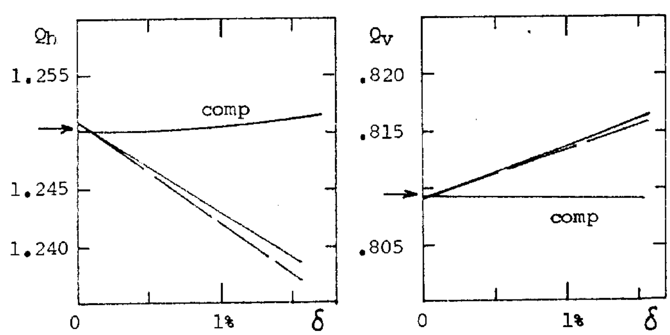

Fig. 3: Horizontal and vertical tunes vs. momentum defect, with and without sextupole chromaticity compensation. Dashed lines: results from COMFORT. Arrows: tunes from linear matrix analysis of lattice.

Fig. 3 shows the dependence of x and y tunes on δ, obtained from a series of runs. It also shows that the chromaticity can be eliminated if desired by adding suitable sextupole coefficients to the gradient magnets. The arrows show that the tunes obtained from the standard matrix analysis of the lattice agree well with those from the nonlinear fit. The dashed lines show the chromaticities obtained by running COMFORT on the same lattice, as was kindly done for us by J. Jäger. We have not tried hard to understand the small discrepancies (12% in Q_h and 4% in Q_v); they could be the difference between a second-order calculation and an exact one. The result for Q_v incidentally illustrates Dragt's comment[3] that the "natural" chromaticity of a small ring can well be positive.

5. Conclusion

This note was intended more to suggest a new way to approach nonlinear effects rather than as a finished product. It is probably useful only for small oscillations; for large ones the fits, though still far better than the linear prediction, are poor. In fact since it is well known that for large oscillations ("far-from-linear" region) the whole concept of a constant tune breaks down, and must be replaced by a whole spectrum of characteristic frequencies, one would not expect anything as simple as (7) to work well. Still, we have here at least a competitive way of finding chromaticities in an arbitrary guide field, and possibly an approach which will give new insights into nonlinear behavior.

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