WIDTH OF NONLINEAR DIFFERENCE RESONANCES

S. Ohnuma

Fermi National Accelerator Laboratory*
P.O. Box 500, Batavia, Illinois 60510

and

R. L. Gluckstern

Department of Physics and Astronomy, University of Maryland
College Park, Maryland 20742

SUMMARY

We consider an isolated difference resonance of the form $(2p)\nu_1 - (2q)\nu_2 = n + \varepsilon$ where $(2p)$ and $(2q)$ are positive integers with $(2p)+(2q)>2$, n is 0 or an integer and $|\varepsilon|<<1$. With action-angle variables (I_k, a_k), the driving term of this resonance in the Hamiltonian takes the form $D\cdot(2I_1)^p(2I_2)^q \cos(\phi)$, $\phi = (2p)a_1-(2q)a_2$ +const. Unlike sum resonances, two action variables I_1 and I_2, which are proportional to emittances in two directions, are bounded and any definition of resonance width will involve the concept of an "acceptable" growth in I_1 or I_2. We propose a definition such that inside the resonance width, an initial condition of large I_2 and very small I_1 will lead to an order of magnitude growth in I_1. With this definition, the width is indefinite for $(2p)=1$. An arbitrarily small I_1 can grow to a sizable fraction of $(p/q)I_2$ for any value of $|\varepsilon|$. For $(2p)=2$, the width is proportional to $D\cdot(I_2)^q$. One cannot have resonances for $(2p)>2$ according to this definition, but there is a threshold value of initial I_1 above which I_1 will grow by a large factor if $|\varepsilon|$ and the invariant quantity $I_1+(p/q)I_2$ satisfy a certain relation which will be given analytically. We thus propose a definition involving one parameter for $(2p)=2$ and two for $(2p)>2$. The picture is clearly symmetric in two directions: if the initial I_2 is very small and I_1 large, one simply uses $(2q)$ in place of $(2p)$ to classify the resonances.

INTRODUCTION

It is well-known[1] that an isolated difference resonance of the form $(2p)\nu_1 - (2q)\nu_2 = n$ does not lead to an instability. The motion is always bounded in both directions and, if πE_1 and πE_2 are emittances in two directions, the quantity $E_1/(2p)+E_2/(2q)$ remains unchanged. ("Emittance" is commonly used to describe a beam as a whole. In this note, we consider each particle to have its own emittance.) This invariant quantity is a manifestation of the exchange of energy from one to the other direction which is familiar in the linear coupling, $(2p)=(2q)=1$. Because of this bounded nature of the motion, one cannot avoid certain arbitrariness in the definition of resonance width. The purpose of this note is to propose one definition in which the concept of an "acceptable" growth in the emittance plays the essential role. The definition will clarify, for example, the physical maeaning of an "infinite" width which results from the Guignard's expression[2] when $(2p)$ or $(2q)$ is unity and E_1 or E_2 approaches zero. The concept of an acceptable growth is introduced here primarily because of its practical importance. Although the motion is bounded, an initially very small emittance in one direction, say E_1, may grow to a large value if $(p/q)\cdot E_2$ is initially very large. For example, in many accelerators, one tries to avoid a growth in the vertical emittance caused by difference resonances when the horizontal emittance happens to be large. Unlike Guignard's definition, widths defined here cannot be expressed analytically for all combinations of p and q but it is easy to evaluate them numerically once the definition is clearly understood. A numerical table will be given for some combinations of p and q which are likely to be of practical interest.

As common in this type of treatment of nonlinear resonances, two approximations are made, one essential and the other not so but nevertheless needed to keep analytical expressions manageable:
1) Only one resonance is considered at a time so that the treatment is best suited when the tunes are close to one particular resonance only. To improve this approximation, one must go to the next order in D which involves a canonical transformation of the action-angle variables. 2) In the action-angle formalism, the Hamiltonian can have terms which are independent of the angle variables. The tune is then a function of the emittances. In deriving the resonance width analytically, one ignores such terms for the sake of simplicity. It is however straightforward to include them for evaluating the width numerically and the invariant expression $I_1+(p/q)I_2$ is unaffected by their presence. As has been discussed extensively by Montague[3] for $(2p)=(2q)=2$, phase-independent terms play a significant rôle when one considers the nonlinear beam-beam interactions in storage rings.

ACTION-ANGLE FORMALISM AND TWO INVARIANTS[4]

For a nonlinear difference resonance of the form $(2p)\nu_1 - (2q)\nu_2 = n + \varepsilon$, (2p) & (2q)=positive integers, n =0 or an integer and $|\varepsilon|<<1$, the resonance-driving term in the Hamiltonian in terms of action-angle variables $(I_k, a_k; k=1,2)$ is

$$D\cdot(2I_1)^p(2I_2)^q \cos(\phi) \tag{1}$$

with $\phi \equiv (2p)a_1 - (2q)a_2 + \text{const.}$ The parameter D is a function of the multipole field

$$c_{N-1} \equiv (1/B\rho)\partial^{N-1}B_y/\partial x^{N-1}|_{x=y=0} \; ; \; N \equiv (2p)+(2q) \tag{2}$$

and the standard linear machine parameters $(\beta_k, \psi_k; k=1,2)$:

$$D \equiv \frac{1}{(2\pi)2^{N-1}(2p)!(2q)!}\left|\int d\ell\, (\beta_1^p\beta_2^q)\, c_{N-1} \times e^{i(2p\,\psi_1 - 2q\,\psi_2 - \varepsilon\theta)}\right| \tag{3}$$

in which the integral is for the entire ring. The independent variable θ is related to the central path length ℓ, $\theta = \ell/(\text{average machine radius})$. Action variable I_k is essentially the emittance πE_k of a particle, $\pi E_k = \pi(2I_k)$. It is convenient to define two dimensionless quantities u^2 and v^2 which are proportional to

*Operated by the Universities Research Association, Inc. under contract with the U.S. Department of Energy.

0018-9499/85/1000-2261$01.00© 1985 IEEE

$(2I_1)$ and $(2I_2)$, respectively,

$$u^2 = \alpha_u (2D/|\varepsilon|)^{1/s}\ (2I_1), \tag{4}$$

$$v^2 = \alpha_v (2D/|\varepsilon|)^{1/s}\ (2I_2), \tag{5}$$

$$\alpha_u = (2p)^{q'/s}(2q)^{q/s}, \tag{6}$$

$$\alpha_v = (2p)^{p/s}(2q)^{p'/s} \tag{7}$$

where $p'=1-p$, $q'=1-q$ and $s = p+q-1$. Note that the quantity D defined by Eq.(3) has the dimension of $(\text{length})^{-s}$. The invariance of $E_1/(2p) + E_2/(2q)$ is equivalent to the invariance of

$$\sigma^2 \equiv u^2 + v^2 . \tag{8}$$

Since the Hamiltonian itself is invariant (independent of the variable θ), one can derive other invariant expressions from linear combinations of the Hamiltonian and σ^2. As the second invariant quantity, we choose

$$\lambda \equiv u^2 + u^{2p}v^{2q}\ w \tag{9}$$

with $w \equiv (\varepsilon/|\varepsilon|)\cdot\cos(\phi)$. For physically meaningful motions, both u^2 and v^2 must be non-negative and w must lie between -1 and $+1$. In the previous report[4] which dealt with sum reonances, w was plotted as a function of u^2 for a fixed value of σ^2, different values of λ giving different curves in (u^2,w) space. Here, as we are interested in the growth in u^2 (which is proportional to $2I_1$) when its initial value is very small, it is more convenient to see u^2 *vs* λ, again for a fixed σ^2. Different curves in (λ,u^2) space correspond to different values of w, the simplest being a straight line $u^2 = \lambda$ for $w=0$. For our purpose, it is sufficient to study two limiting curves, one for $w=+1$ and the other for $w=-1$. Typical behaviors are illustrated in Figs.(A)-(D).

RESONANCE WIDTH

1. $\boxed{(2p) = 1}$

According to Guignard, the width is

$$\Delta e \equiv 2|\varepsilon| = 2D\cdot(E_2)^q/\sqrt{E_1} \tag{10}$$

for $E_1 \ll E_2$ and this grows indefinitely as E_1 approaches zero. The behavior in (λ,u^2) space shown in Fig.(A) is valid for any value of $|\varepsilon|$ and the physical meaning of an indefinitely growing width is clear from this. An arbitrarily small u^2(i.e., E_1) can grow to a sizable fraction of the maximum possible value, $\sigma^2 \simeq v^2$(initial). For $v^2(\text{initial}) \ll 1$, the emittance E_1 can increase to values at least as large as

$$(2D/|\varepsilon|)^2 (E_2)^{2q}_{\text{initial}} \tag{11}$$

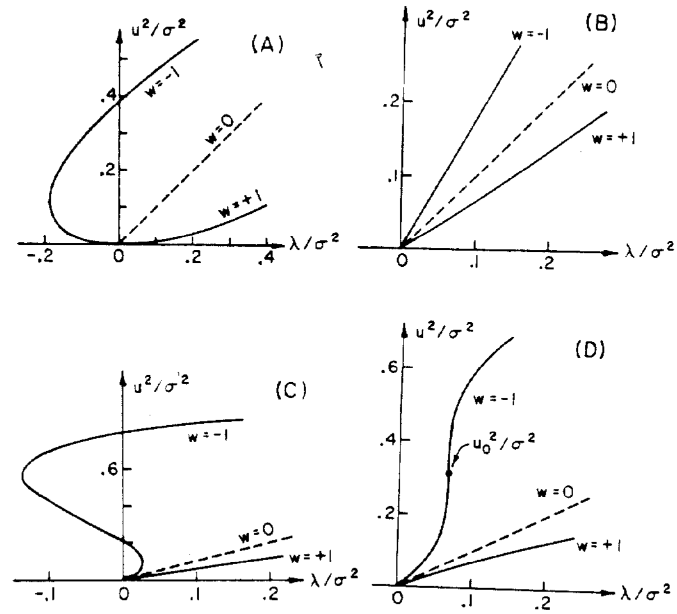
For $v^2(\text{initial})=2$, E_1 can become as large as one-half of the mzximum possible value $(p/q)E_2$(initial) for any q, i.e., if $\sigma^2=2$, then $u^2=1$ for $\lambda=0$ and $w=-1$.

2. $\boxed{2p = 2}$

The physical interpretation of our definition is straightforward for resonances of this type. When the initial value of E_2 is sufficiently small, a small E_1 stays proportionately small as illustrated in Fig.(B). Beyond a certain threshold value, the behavior changes into the one of Fig.(A) so that an initially very small E_1 can grow to a large value as for the previous case, $(2p)=1$. The threshold condition

$$\sigma^{2q} = 1 \tag{12}$$

can be written in the form



For (2p)=1, (A) is applicable under any condition. For (2p)=2, the picture changes from (B) to (A) as one moves from outside to inside the resonance width. For (2p)>2, the change is from (B) to (D) to (C) as $|\varepsilon|$ decreases or the initial emittance $E_1+(p/q)E_2$ increases.

$$|\varepsilon_0| = 4D\cdot(E_2)^q_{\text{initial}} \tag{13}$$

showing the relation between the resonance width $|\varepsilon_0|$ and the initial emittance. In deriving Eq.(13) from Eq.(12), it is assumed that the initial value of E_1 is much smaller than E_2(initial) so that $\sigma^2 \equiv (u^2+v^2)_{\text{init.}} \simeq v^2$(initial). This is justified since we are interested in the possible growth of E_1 starting from a very small value. For a given E_2(initial), if $|\varepsilon|$ is less than $|\varepsilon_0|$ (inside the resonance), the emittance E_1 can grow at least to the value

$$\frac{1}{q}\ (E_2)_{\text{initial}} \times (1 - |\varepsilon/\varepsilon_0|^{1/q}) \tag{14}$$

corresponding to $u^2 = \sigma^2 - 1$ for $\lambda=0$ and $w=-1$. This quantity approaches the maximum possible value $(p/q)E_2$ as ε approaches zero. Comparing our definition, Eq.(13), with the expression given by Guignard, we find that our width is exactly twice as large. The fact that this ratio two is equal to $(2p)$ is not accidental. As one can see in *ref. 4*, the argument based on the concept of "fixed lines" in (I_k,a_k) space leads to a factor $(2p)^2$ in the expression of width. Indeed, this factor appears in Guignard's formula for the width of sum resonances (*ref. 2*, p.76) but the corresponding factor is $(2p)$ in his definition of difference resonance width.

3. $\boxed{(2p) > 2}$

Complex features of the resonance belonging to this class are shown in Figs.(B)-(D). One notices that in all these pictures, if the initial value of u^2 is sufficiently small, it remains small in a proportionate manner. In this sense, there is no resonance according to our definition. However, in Fig.(C), there is a threshold value of initial u^2 above which u^2 will grow by a large factor as in Fig.(A). Fig.(D) shows the situation when the character of the coupled motion changes qualitatively from that in (B) to the one in (C). One may thus modify the strict definition which was used for (2p)=1 and 2, and derive the relation between $|\varepsilon|$ and the invariant emittance $E_1 + (p/q)E_2$ in Fig.(D).

In order to find the inflection point u_o^2 in Fig.(D) and the corresponding value of σ^2, one must solve the following three equations simultaneously,

$$d\lambda/d(u^2) = 0, \quad d^2\lambda/d(u^2)^2 = 0, \quad w = -1 \qquad (15)$$

where λ is given by Eq. (9). The algebra is elementary but rather messy. The solution is

$$(\sigma_o^2)^s = \sqrt{s/(pq)}\ (u_o^2/\sigma_o^2)^{p'} (v_o^2/\sigma_o^2)^{q'}; \quad p'=1-p,\ q'=1-q \qquad (16)$$

with

$$(u_o^2/\sigma_o^2) = -\, p'/(s + \sqrt{sq/p}\,), \qquad (17)$$

$$(v_o^2/\sigma_o^2) = -\, q'/(s - \sqrt{sp/q}\,) \quad \text{for } q \neq 1,$$
$$= 2/(1+p) \quad \text{for } q = 1 \qquad (18)$$

One sees that $u_o^2 + v_o^2 = \sigma_o^2$ as it should be. In order to find the resonance width $|\varepsilon_o|$ which corresponds to Fig.(D), one evaluates σ_o^{2s} from Eqs.(16)-(18) and use the relation

$$|\varepsilon_o| = (\alpha_u^s/\sigma_o^{2s})(2D)(E_T)^s \qquad (19)$$

where the invariant emittance $E_T = E_1 + (p/q)E_2$ should be very close to $(p/q)(E_2)_{init}$. As the threshold value of u^2 above which it can grow by a large factor, one is tempted to use the analytic expression Eq.(17), but this will be an overestimate. Rather, it should be u^2 lying on the curve w=+1 (call it u_+^2) sharing the same value of λ with u_o^2 on the curve w=-1. It will be the solution of

$$u_+^2 + u_+^{2p}(\sigma_o^2 - u_+^2)^q = u_o^2 - u_o^{2p}v_o^{2q}\ . \qquad (20)$$

The corresponding value for E_1, $(E_1)_{thr}$, is

$$(E_1)_{thr} = (u_+^2/\sigma_o^2)\cdot E_T \qquad (21)$$

with the same E_T as in Eq.(19). Unfortunately, it is not possible to express u_+^2 analytically; Table 1 lists numerical values of (u_+^2/σ_o^2) as well as of $(\alpha_u^s/\sigma_o^{2s})$, the factor appearing in Eq.(19), for low-order resonances.

Eqs.(19) and (21) together with numerical values listed in Table 1 specify the threshold condition completely. A natural question to follow is: what is the relation between E_T and $(E_1)_{thr}$ when one is inside the resonance, i.e., if σ^2 is larger than σ_o^2 ? This is the case illustrated in Fig.(C). The point corresponding to u_o^2 of Fig.(D) now satisfies only two conditions,

$$d\lambda/d(u^2) = 0, \quad w = -1\ . \qquad (15')$$

Once this point is found, one evaluates the corresponding value of λ and, to find $(E_1)_{thr}$, u_+^2 must be found from

$$u_+^2 + u_+^{2p}(\sigma^2 - u_+^2)^q = \lambda \qquad (20')$$

Table 1. Numerical Factors in Eqs. (19) & (21)

(2p)	(2q)	$(\alpha_u^s/\sigma_o^{2s})$	(u_+^2/σ_o^2)
3	1	1.02	.049
	2	.89	.029
	3	1.13	.021
	4	1.76	.016
4	1	1.07	.14
	2	.67	.089
	3	.63	.067
	4	.77	.054
5	1	1.16	.22
	2	.56	.16
	3	.43	.12
	4	.42	.099

Clearly it is not possible to have analytical solutions for general combinations of p and q but the numerical evaluation is not difficult. We will simply mention two qualitative features:

1) If σ^2 is close to σ_o^2, the change in u_+^2 is shown to have an approximate dependence

$$\Delta(u_+^2/\sigma^2) \propto -\sqrt{\Delta(\sigma^2)} \qquad (22)$$

2) For σ^2 not too close to σ_o^2, the relation

$$(E_1)_{thr}^{p-1}\cdot(E_T - E_1)^q \simeq \text{const.} \qquad (23)$$

seems to be valid for a large range of σ^2/σ_o^2. This relation is suggested by the concept of "fixed lines" in (I_k, a_k) space[4] and is used by Guignard also.

Finally, it may be natural to consider the distribution of particles as a function of two invariants, λ and σ^2, instead of more common (u^2, v^2). If the initial distribution is given as $f(u^2,v^2)du^2dv^2$ (assuming no phase dependence), one can derive the corresponding distribution $F(\lambda,\sigma^2)d\lambda d\sigma^2$. However, in view of the simplifying approximations made, this may not be too useful in practical situations.

References

1. P. A. Sturrock, Annals of Physics, 3, 113(1958).
2. G. Guignard, *Selection of Formulae Concerning Proton Storage Rings*, CERN 77-10, 6 June 1977.
3. B. W. Montague, Nucl. Inst. Methods, 187, 335(1981).
4. Notations used here are identical to those in the previous report on sum resonances: S. Ohnuma, IEEE Trans. Nucl. Sci., NS-28, 2491(1981). See also S. Ohnuma, UM HE 84-1 (University of Michigan), Report of a workshop on SSC, p. 149.