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APPLICATION OF DYNAMICALLY CONSISTENT CLOSURES TO HYDRODYNAMIC MODELS OF BEAMS*

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For applications to phenomena such as pinched beams in ion beam fusion or beams in dominant axial magnetic fields (ion sources, plasma lens etc.), in accelerators, dynamically consistent closure of hydrodynamical models is discussed within adiabatic assumptions mathematically analogous to that of Chew, Goldberger, and Low[1] (abbrev. CGL) for magneto-hydrodynamics and that of Berman and Mark[2] (abbrev. BM) for galactic dynamics. Numerical comparison with particle codes is discussed for pinched beams.

Hydrodynamic models have been used in particle beam research.[3,4] Sometimes equations of state are assumed by dropping heat fluxes[5] without detailed attempt to model possible beam behavior such as anisotropic stresses even in the transverse directions. Recently, for the purpose of ion beam inertial fusion research,[6,7] we have shown that dynamically consistent equations of state can be derived from the Vlasov equation for some beam models under assumptions stated below: The particle orbits are allowed to be general rosette orbits in central fields, except that the ratio of maximum to minimum radial excursion is limited by something less than a factor of two. If the beam has zero net angular momentum, then consider at least two counter-rotating beam components of this type. By remaining closely tied to the actual particle orbits, we retain the possibility of wave-particle resonances typical of resistive-hose instabilities. It is not clear that other recent attempts[8] at fluid models have retained such resonances. In terms of the usual cylindrical coordinate system (r, θ, z) with beam axis in the z direction, the above and other adiabatic assumptions are summarized by

$$\varepsilon = O\left(\frac{\delta r}{r}\right) = O\left(\frac{rp_r}{p_\theta}\right) = O\left\{\left[\frac{\partial}{\partial t} + (\dot{z} - v_{zp})\frac{\partial}{\partial z}\right]/\nu\right\} = O\left(\frac{\partial}{\partial\theta}\right) < 1 \quad , \tag{1}$$

where δr is the deviation from average radius in the above mentioned particle orbit, (p_r, p_θ, p_z) are the particle momenta, t is the time coordinate, and ν is the radial oscillation frequency of the rosette orbits (to lowest order, see Eq. 5 below). Dots over symbols indicate time derivatives. The derivation assumes some frame of velocity v_{zp} where $[\partial/\partial t + (\dot{z} - v_{zp})\partial/\partial z] = O(\varepsilon)$ which does not need to have the velocity of either the beam, a wave packet disturbance or the laboratory. The final results could be transformed to other frames. Assumptions (1) will allow non-linear waves: for example, for non-axisymmetric nonlinear disturbances in the distribution function, a hypothetical $F[H, \varepsilon b(r,\theta,z,p_r,p_\theta,p_z,t)]$ is consistent with (1), where H is the Hamiltonian of a moving-beam equilibrium (independent of θ). We also assume for simplicity that the dominant external force on a beam particle of charge q is the radial force $f_r = q(E_r - \dot{z}\,B_\theta/c)$ where E_r is the radial electric field and B_θ is the θ-component of the magnetic field, and c is the speed of light. Generalizations to additional central forces or relativistic beams are straightforward. Non-radial force components are assumed to be of higher order in ε.

The mathematical framework is initiated by transforming Vlasov's equation to a different coordinate system in momentum space,

$$\dot{r} = U(r, \theta, z, \dot{z}, t) + \alpha\sin\Psi \quad , \tag{2}$$

$$r^2\dot{\theta} = r^2\,\Omega(r, \theta, z, \dot{z}, t) + \beta\cos\Psi \quad . \tag{3}$$

Since the "drifts" U and ($r\Omega$) can be identified with low approximations of the hydrodynamic motions, the analogue of "gyro-motion" is relegated to the rapid ($\alpha\sin\Psi$) and ($\beta\cos\Psi$) quantities with fast gyration frequency $\nu = \dot{\Psi}$. Here,

$$\Omega^2(r, \theta, z, \dot{z}, t) = -\frac{1}{rm}f_r = \frac{q}{mr}\left(\frac{\dot{z}}{c}\,B_\theta + E_r\right) \quad , \tag{4}$$

$$\nu^2(r, \theta, z, \dot{z}, t) = \frac{1}{r^3}\frac{\partial}{\partial r}\left[r^4\,\Omega^2\right] \quad , \tag{5}$$

$$U(r, \theta, z, \dot{z}, t) = -\frac{2\Omega}{r\nu^3}\left[\frac{\partial(r^2\Omega)}{\partial t} + \Omega\,\frac{\partial r^2\Omega}{\partial\theta} + \dot{z}\,\frac{\partial r^2\Omega}{\partial z} - \frac{1}{m}f_\theta\right] \quad , \tag{6}$$

and the ratio of amplitudes

$$\frac{\beta}{\alpha} = \frac{r\nu}{2\Omega} \quad , \tag{7}$$

where m is the mass of the beam particle, and f

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is the body force. Following CGL we expand $F = \sum_{j=0}^{\infty} \epsilon^{j-2} F_j$ which begins with $O(\epsilon^{-2})$ because the spatial number density $n = O(1)$ in ϵ. Under our adiabatic assumptions it is straightforward to derive from the transformations (2)-(3) the "drift-kinetic" equation to first order in ϵ,

$$\frac{\partial F_o}{\partial t} + U \frac{\partial F_o}{\partial r} + \Omega \frac{\partial F_o}{\partial \theta} + \dot{z} \frac{\partial F_o}{\partial z} + \ddot{z} \frac{\partial F_o}{\partial \dot{z}} = 0 \ . \quad (8)$$

Here we have utilized the fact that the distribution function $F_0(r, \theta, z, \dot{z}, \mu, t)$ is independent of the rapid-phase Ψ, a fact which follows because the zeroth order (in ϵ) form of Vlasov's equation is $\partial F_0/\partial \Psi = 0$ as in CGL theory. Also, we have simplified the drift-kinetic equation (8) by taking advantage of the existence of an adiabatic invariant μ expressed to lowest significant order in ϵ as

$$\mu = \frac{2}{\nu} \left\langle (\dot{r} - U) \right\rangle^2 \ , \quad \frac{d\mu}{dt} = 0 \ . \quad (9)$$

The relation to hydrodynamic motions follows from deriving equations for the stresses, which are defined in terms of the averages

$$\langle h \rangle_F \equiv \int h\, F \frac{\nu^2}{4\Omega n}\, d\mu\, d\Psi\, d\dot{z} \ . \quad (10)$$

For example, the number density of particles in this beam coordinate is

$$n \equiv \int F \frac{1}{r}\, dp_r\, dp_\theta\, d\dot{z} = \int F_o \frac{\nu^2}{4\Omega}\, d\mu\, d\Psi\, d\dot{z} \ . \quad (11)$$

To lowest significant order in ϵ, the components P_{rr}, $P_{r\theta}$ etc. of the stress tenser is related to F_0 by

$$P_{rr} = n \left\langle \mu\nu \sin^2\Psi \right\rangle_{F_o} = \frac{1}{2} n \langle \mu\nu \rangle_{F_o} \ , \quad (12)$$

$$P_{\theta\theta} = n \left\langle \mu \frac{\nu^3}{4\Omega^2} \cos^2\Psi \right\rangle_{F_o} \ , \quad (13)$$

$$P_{r\theta} = P_{\theta r} = n \left\langle \frac{\mu\nu^2}{4\Omega} \sin 2\Psi \right\rangle_{F_o} = 0 \ , \quad (14)$$

$$P_{rz} = P_{zr} = n \left\langle (\mu\nu)^{1/2} \sin\Psi\, (\dot{z} - v_z) \right\rangle_{F_o} = 0 \ , \quad (15)$$

$$P_{\theta z} = P_{z\theta} = n \left\langle \frac{\mu^{1/2}\nu^{3/2}}{2\Omega} \cos\Psi\, (\dot{z} - v_z) \right\rangle_{F_o} = 0 \ , \quad (16)$$

$$P_{zz} = n \left\langle (\dot{z} - v_z)^2 \right\rangle_{F_o} \ . \quad (17)$$

The continuity and momentum hydrodynamic equations of relevance can be written for each rotating beam component

$$\frac{Dn}{Dt} + n \left[\frac{1}{r} \frac{\partial (r v_r)}{\partial r} + \frac{1}{r^2} \frac{\partial j}{\partial \theta} + \frac{\partial v_z}{\partial z} \right] = 0 \ , \quad (18)$$

$$\frac{Dv_r}{Dt} = \frac{j^2}{r^3} + \frac{1}{m} f_r - \frac{1}{mn} \left[\frac{\partial P_{rr}}{\partial r} + \frac{1}{r} (P_{rr} - P_{\theta\theta}) \right] , \quad (19)$$

$$\frac{Dj}{Dt} = \frac{r}{m} f_\theta - \frac{1}{mn} \frac{\partial P_{\theta\theta}}{\partial \theta} \ , \quad j \equiv r v_\theta \ , \quad (20)$$

$$\frac{Dv_z}{Dt} = \frac{1}{m} f_z - \frac{1}{mn} \frac{\partial P_{zz}}{\partial z} \ , \quad (21)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + \frac{j}{r^2} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \ . \quad (22)$$

In equation (9), the frequency ν plays the role of $(q B_z/mc)$ of the CGL theory. A physically suggestive "field equation" for ν is

$$\frac{D}{Dt} \left(\frac{\nu^2 r^2}{n\, |J|} \right) = 0 \ , \quad (23)$$

which can be derived from equations (4)-(6), and (18) provided $\dot{z}$ does not vary much over local beam elements. Closure in terms of generalized adiabatic equations of state is possible if in addition $|\dot{z} - \langle \dot{z} \rangle| / \langle \dot{z} \rangle = O(\epsilon^2)$ and $(\partial/\partial z) = O(\epsilon^2)$ (for example, negligible longitudinal Landau damping). We find under these conditions that

$$P_{\theta\theta} = \frac{r^4 \nu^2}{4 j^2} P_{rr} \ , \quad (24)$$

$$\frac{d}{dt} \left(\frac{P_{rr}}{n\nu} \right) = 0 \ , \quad \frac{d}{dt} \left(\frac{P_{zz}}{n} \right) = 0 \ . \quad (25)$$

For the best fully hydrodynamic model, closure equations (23)-(25) should be used with the continuity and momentum equations (18)-(22) which are exact velocity moments of Vlasov's equation.

An axisymmetric particle code simulation with 3000 particles was performed for a Gaussian-profile electron beam with initial radius of 1 cm, beam energy 50 MeV and current 1 kA. The beam is assumed to be fully space charge neutralized but with no current neutralization. Since a particle code effectively calculates the full time development of the spatial and velocity distribution functions, phase mixing of beam particle trajectories is depicted without further approximation, except for numerical ones such as numerical noise. A beam initially at equilibrium oscillates radially when perturbed.

The dashed curve in Fig. 1 represents the time development of the RMS beam radius, showing phase-mix damping of the oscillations. The above beam-hydrodynamics model was used in a Lagrangian fluid code to recalculate the situation simulated by particle code above. The hydrodynamic model result is shown as the solid curve in Fig. 1. It is evident that phase mixing is well reproduced. The apparent phase

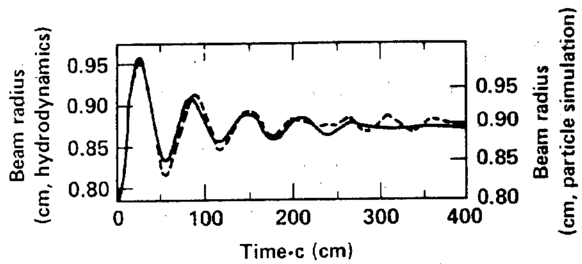

Fig. 1. Phase-mix damping of radial oscillations.

difference can be further adjusted by a slight re-alignment of scale. It is difficult to produce identical initial conditions for the two calculations. The residual oscillation of the particle code is due to finite particle number. Fifty radial zones were used in the hydro-code calculation; artificial viscosity was introduced for numerical stability of large radial perturbations. The initial amount of angular momentum assigned to each fluid element of the hydrodynamic model can be varied somewhat. It has been found that realistic radial behavior is obtained if the fluid element angular momentum is a linearly increasing function of radius. The average centrifugal force is 1/2 of the average pinch force.

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