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# A PROBABILITY FUNCTION TO FIT RADIAL DISTRIBUTIONS IN PARMILA* SIMULATION BEAMS†

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## Summary

Proton or deuteron beams in linear accelerators have halos much too large to be described by the normal distribution. When the beam core is fitted to a normal distribution, the actual amount of beam past a given number of standard deviations is orders of magnitude above the normal distribution prediction. Knowing the distribution obeyed by beams would be helpful in evaluating beam spill using computer codes. Toward this end, a probability distribution has been found that fits PARMILA-generated radial beam distributions. The fitting distribution is a mixture of two generalized gamma (gg) distributions and fits both the beam core and the tail reasonably well. Its predictions in the halo region are substantially larger than normal distribution predictions. The distribution is described, examples of fits are given, and the ability to fit the tail is discussed.

## Introduction

The type of radial distribution obeyed by a linac beam is of interest for several reasons. The normal distribution often is used to describe the beam even though it is known that a physical beam does not obey a normal distribution in the transverse coordinates. In particular, the number of halo particles is much too large for the distribution to be normal. Further, calculations (space-charge forces for instance) often are based on an assumed particle distribution in the beam, and it would be helpful to know how well or how poorly the assumed distribution fits the beam. An application of more immediate interest has been to find a distribution that fits the radial particle distribution in a PARMILA simulation beam for use in predicting beam spill. Beam-spill prediction in linacs is becoming more important as higher current linac designs are being proposed. Once an acceptable probability distribution has been found, it overcomes the beam-spill prediction problems caused by poor resolution that is due to the limited number of pseudoparticles in the simulation beam.

Why find a fit to the distribution produced by a computer code? Because the code is much cheaper and easier to work with than an actual beam. Moreover, the fitting must be done to the computed distribution when the distribution is used for predictive purposes. Using a code means that results will apply to a real beam only to the extent that the pertinent physics is in the code. It is generally conceded that PARMILA models actual beams fairly well; certainly it does well on the beam cores. Thus there is hope that distribution results will be applicable to reality.

It seems a waste of time to find a distribution that fits all the particles if one is merely interested in the tails. However, there are few particles in the tails and, especially since particles are moved into the tails by the same code physics that forms the core, a consistent distribution describing all the particles contains more information and can be trusted more than an *ad hoc* fit of the tail alone.

*PARMILA is a multiparticle simulation code in which each particle is transported through the accelerator elements using the chain-matrix method. The treatment is fully 6-D, and the code contains highly detailed, nonlinear modeling.

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## Proposed Fitting Distribution

The proposed fitting distribution is a mixture of two gg distributions. The gg distribution[1-3] is a probability distribution of great generality and is a three-parameter function whose density is given by

$$f(r) = \gamma\left(\frac{\alpha}{\beta}\right)^{\alpha} \frac{r^{\gamma\alpha-1}}{\Gamma(\alpha)} e^{-\left(\frac{\alpha}{\beta}\right)r^{\gamma}} \quad ; \ \alpha,\ \beta,\ \gamma > 0;\ r \geq 0 \quad . \ (1)$$

This distribution contains many other distributions as special cases, among which are the exponential, gamma, chi, chi-square, Weibull, Rayleigh, and Maxwell distributions. Figure 1 shows the distribution shape for several values of the $\gamma$ parameter. Obviously the distribution can take on a fair range of shapes since it changes continuously as $\gamma$ is varied.

A mixture distribution was chosen only after a broad class of pure distributions had been tried and rejected.[3] In particular, the binormal distribution, commonly assumed to describe the beam, was rejected because of its poor fit. (The binormal distribution is the radial distribution the particles would obey if they had independent normal distributions in both transverse coordinates, which is the common assumption.) Figure 2 shows a typical fit of a binormal distribution to a PARMILA radial distribution. The binormal distribution always fails to reach the proper peak height, lies above the particle distribution between the peak and the tail, and falls well below the particle distribution in the tail region.

The two criteria used for judging the fit were a standard chi-square goodness-of-fit test and a comparison of the counted number of particles past a given radius (usually 1 cm) to the number predicted there. The second criterion obviously is the more important when the distribution is used to study beam spill.

Of all the pure distributions, only the gg distribution came close to fitting the particle distribution consistently; Fig. 3 shows a typical example of a

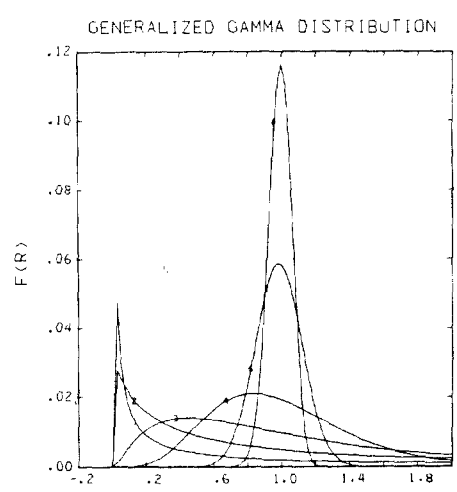


Fig. 1. Shapes assumed by the gg distribution. For all curves $\alpha = 6$, $\beta = 1$. Curve 1, $\gamma = 0.1$; Curve 2, $\gamma = 0.25$; Curve 3, $\gamma = 0.5$; Curve 4, $\gamma = 1$; Curve 5, $\gamma = 3$; Curve 6, $\gamma = 6$.

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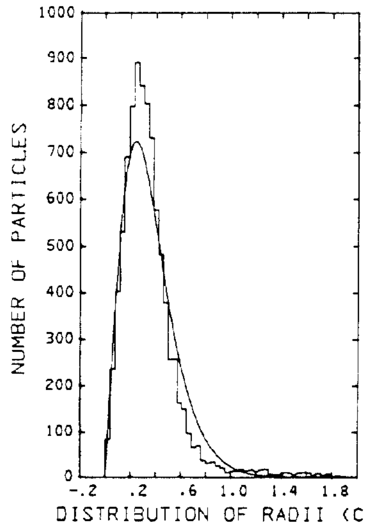


Fig. 2. The fitted binormal distribution (curve) provides a poor fit to the PARMILA radial particle distribution (histogram).

pure gg distribution fitted to a PARMILA radial distribution. Unfortunately the gg distribution only rarely meets the chi-square goodness-of-fit criterion when it is applied to the PARMILA beam, and a comparison of the predicted and counted tail contents shows that it tends to underestimate the tail consistently. The example (Fig. 3) fails the chi-square criterion by a substantial margin and predicts only 32% of the actual tail content.

In the example and in other fits, failure of the gg distribution to fit the particle distribution often seems to be due to the particle distribution's having too broad a peak region and too heavy a tail. Both problems can be solved by using a mixture distribution, provided one of the mixture-distribution components has a nonzero location parameter.

In the mixture distribution, Eq. (2), the origin of r is taken to be the beam centroid. The location parameter, $\lambda$, is the radius about the centroid inside which there are no particles belonging to Group II in Eq. (2); quantities p and q are ratios of the numbers in Groups I and II to the total number.

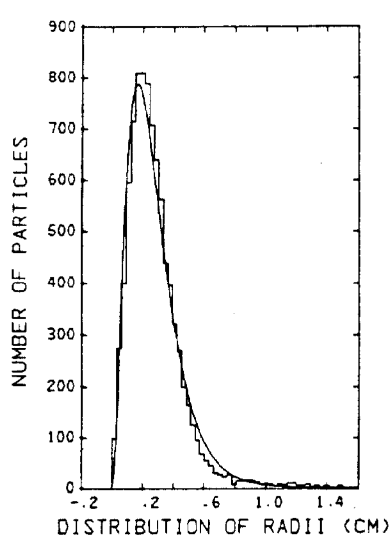


Fig. 3. The gg distribution provides a better fit to the PARMILA radial distribution than any other pure distribution tried.

$$f(r) = p\left(\frac{\alpha_I}{\beta_I}\right)^{\alpha_I} \frac{r^{\gamma_I\alpha_I-1}}{\Gamma(\alpha_I)} e^{-\left(\frac{\alpha_I}{\beta_I}\right) r^{\gamma_I}} + q\left(\frac{\alpha_{II}}{\beta_{II}}\right)^{\alpha_{II}} \frac{(r-\lambda)^{\gamma_{II}\alpha_{II}-1}}{\Gamma(\alpha_{II})} e^{-\left(\frac{\alpha_{II}}{\beta_{II}}\right)(r-\lambda)^{\gamma_{II}}} . \quad (2)$$

Reference 3 describes the procedure used to fit this distribution to the PARMILA radial distribution. The location parameter arbitrarily is set at 1.3 times the mode of the particle distribution, and an iterative procedure separates the two groups.

The mixed-distribution fits are much better than those obtained for pure distributions. In a typical run through the Fusion Materials Irradiation Test (FMIT) linac, about two-thirds of all fits are good by chi-square criterion; the remainder usually are only slightly outside the chi-square acceptance limit. In cases where the fit fails the chi-square test, the reason often seems to be a poor fit in the peak region caused by a jagged particle distribution (Fig. 4). In Fig. 4 the predicted number of particles past 1 cm matched the count exactly.

A second reason for failure to meet the chi-square criterion appears to be a failure of the fitted distribution to reach the peak of the particle distribution. When this occurs, it seems to be for cases when the particles are grouped more strongly than usual toward the centroid. Here the failure may be due to the rather arbitrary choice of the location parameter, and a choice that reduces the location parameter in proportion to the peakedness of the particle distribution might produce a better fit.

The fit is quite good when judged by the tail-counting criterion (Fig. 5). This figure shows the ratio of the number predicted to the number counted

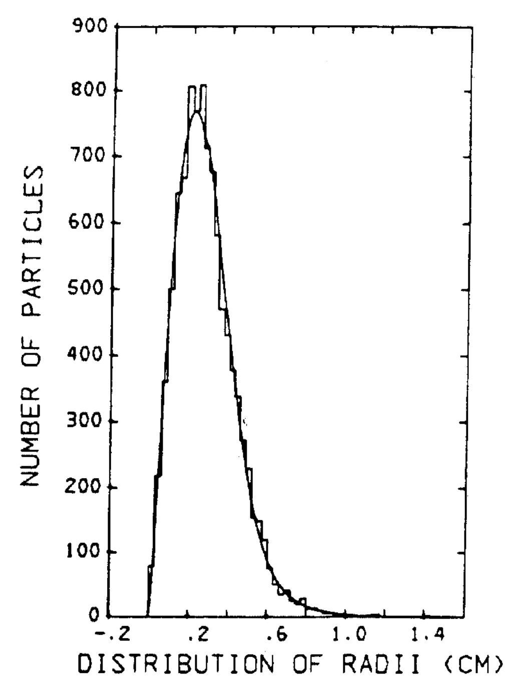


Fig. 4. Failure of the fitted mixed gg distribution to satisfy the chi-square criterion often seems caused by jaggedness in the peak region.

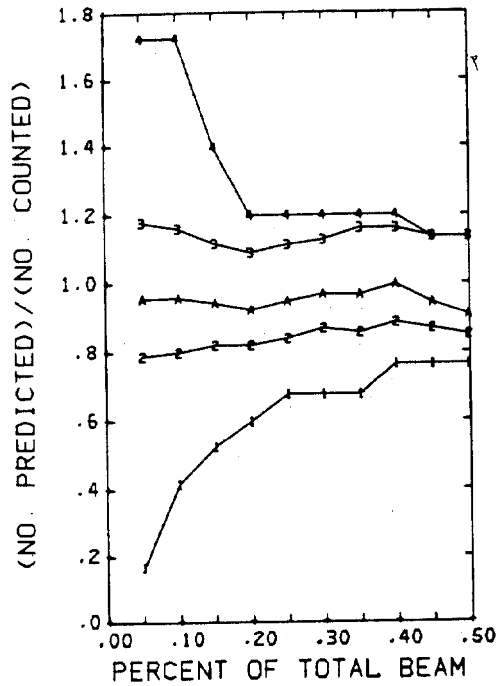


Fig. 5. Ratio of predicted number to counted number of particles past 1 cm versus percent of beam past 1 cm. See text for explanation.

as a function of the percent of the beam past a fixed radius for one run through the FMIT drift-tube linac. Curve A gives the average of all qualifying cells, a prediction that is a little low. Curve 2 is the average value of all low predictions; Curve 3 is the average value of all high predictions. Thus, when the prediction is low, it averages <20% low; when it is high it averages <20% high. Curves 1 and 4 give the worst under- and overestimates seen. As expected, the extremes get worse as the amount of beam in the tail gets smaller. In this example, 0.1% corresponds to <10 particles; therefore, although the worst-case ratios differ substantially from 1.0, they do not represent a large absolute number of particles.

A mixture of two distributions should not, and does not, always lead to a smooth composite profile; Fig. 6 shows a discontinuity in the peak region. When

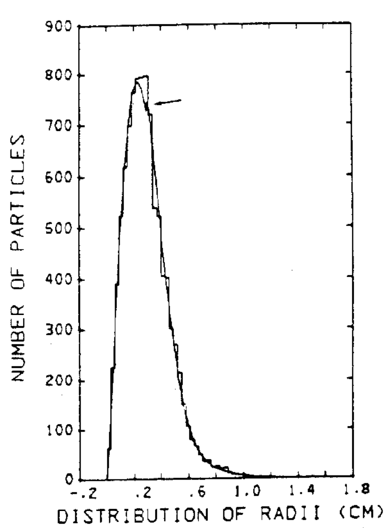


Fig. 6. Example of a discontinuity that occasionally occurs because of using only two distributions in the mixture.

such discontinuities occur, they seem caused by a need to increase the width of the fitted peak and are related to the mixture-distribution's ability to provide a better fit. Using a larger number of component distributions (if it were possible to make the fit) would probably eliminate such discontinuities.

## Conclusions

I have described a mixture distribution that fits the PARMILA-generated radial particle distribution better than any of the common pure distributions. The fits are usually good, judged both by the chi-square and tail-counting criteria. In the preceding I may have emphasized the more pathological aspects of a failure to fit. Figure 7 is an example more typical of the average fit. It is good by both criteria.

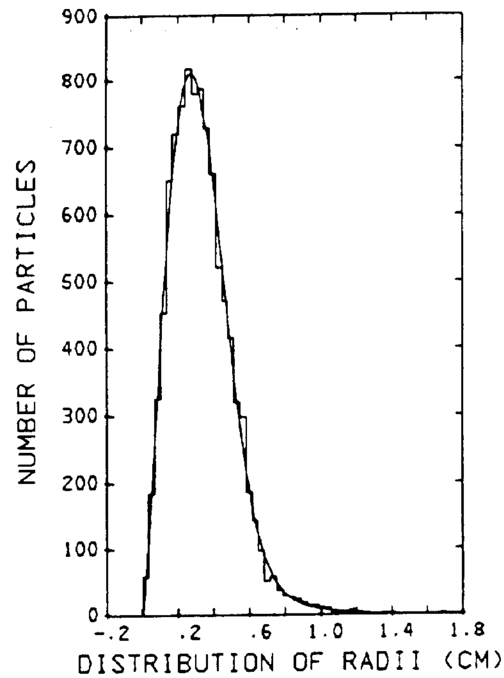


Fig. 7. Typical good mixed gg fit. Of 9703 particles in the simulation beam, 40 were past a 1-cm radius. The fitted distribution predicted 39 particles.

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