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COMPARISON OF DELTA FUNCTION AND GAUSSIAN WAKES FOR LONGITUDINAL TRACKING

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Summary

The longitudinal motion of bunched beams in large storage rings has been simulated in synchrotron phase space, including the effects of wake-fields excited by the bunches themselves in the RF cavities. Using the delta-function wake-potential of the cavities requires a great deal of computing time and, in addition, the results depend on the number of superparticles considered. The computing time can be strongly reduced by using tables of wake potentials of a fixed bunch shape.We investigate the case where the distribution is assumed to be Gaussian. The validity and limitations of this model as function of the beam intensity are analysed by evaluation of higher moments of the distribution (variance, skewness, kurtosis). This shows that the assumption of the shape remaining Gaussian is insufficient and yields misleading results when the beam current is too high. This phenomenon appears to be related to the manifestion of turbulence in the beam.

1. Introduction

This paper compares two approaches to the simulation of bunch lengthening and widening in an electron storage ring. They differ in the description of the wake potential, i.e. the decelerating force which is built up while the bunch traverses an RF cavity and reacts back on later particles in the same bunch.

One way of describing this wake potential is by a sum of wake potentials for point charges (Green's function) which may be obtained by adding the contributions of all the cavities modes[1]. However, this approach requires much computing time and thus conflicts with the legitimate purpose of the simulation, namely the simulation of the bunch behaviour over several damping times. This problem becomes especially evident when the damping time corresponds to a large number of turns. Other representations of the wake potential require much less computing time for the simulation. Here we consider the simplest model: the wake potential for the whole bunch is a function of only two variables, the r.m.s. bunch length, and the time relative to the bunch centre. The aim of this study is not only to validate a model, which we expect to be the fastest, but also to try to shed some light on the relevant parameters for this type of approximation.

2. Theory and models considered

In the simplest case, the storage ring has just one RF station. The longitudinal motion (energy deviation ε, time delay t with respect to the synchronous particle) of the n^{th} particle, evaluated at the m^{th} turn is described by[2,3]:

$$\varepsilon_n^m = \varepsilon_n^{m-1} - \frac{2T_0}{\tau_\varepsilon}\varepsilon_n^{m-1} + U_{BL}^{m-1} + 2\sigma_{\varepsilon_0}\sqrt{T_0/\tau_\varepsilon}\, R_n^m \quad (1)$$
$$+ eV_{RF}\sin(\phi_s + \omega_{RF} t_n^{m-1}) - U_0 + e\, W_n^{m-1}(t_n)$$

$$t_n^m = t_n^{m-1} + \alpha\frac{T_0}{E_0}\varepsilon_n^m \quad (2)$$

where U_{BL} represents the transient beam-loading, T_0 the revolution time, τ_ε the damping time, V_{RF} the applied RF voltage with stable phase ϕ_s and angular frequency ω_{RF}, U_0 the radiation loss per turn, $\sigma_{\varepsilon 0}$ the natural energy spread and α the momentum compaction. R_n is a random number used to describe quantum fluctuations. The main problem of the simulation is hidden in the wake potential W_n.

One method consists of writing the wake potential in terms of the delta-wake function (Green's function) $w(\Delta t)$ - Δt being the difference of position in time between two particles - as:

$$W_n^m(t_n) = \frac{I_0 T_0}{N}\sum_j^{t_n>t_j} w(t_n^m - t_j^m) \quad (3)$$

where N is the number of superparticles, and I_0T_0/N the charge of one superparticle. Note that this sum is subjected to the causality condition $t_n>t_j$.

Unfortunately this approach has two limitations: firstly it becomes very time consuming with increasing number of particles N as the number of related operations scales with the square of N. This can be only partly reduced by grouping (or "binning") a certain number of neighbouring particles[1]. Secondly the results obtained with such a formalism are sensitive to the number of particles considered[1], which usually implies several runs with different N for the evaluation of asymptotic values.

Another method is to approximate the particle distribution by some known functions for which precomputed tables of wake potentials can be prepared. With this approach, the running time scales linearly with N and the results can be considered as asymptotic.

In this paper we investigate the simplest model where we apply - whatever the actual bunchshape is - the wake potential of a Gaussian bunch with the corresponding bunchlength. In other words, we simply state that - as far as the wake potential is concerned - the bunch is fully characterised by the evaluation of the first two moments of the distribution (mean value and r.m.s. standard deviation), which requires a single set of tables of wake-potential for different bunchlengths according to:

$$W_n^m(tn) = W(\sigma^m, t_n - \langle t\rangle) \quad . \quad (4)$$

The same assumption of constant bunchshape is usually made in the analytical approach of the mode-coupling model[4].

The validity of this approximation is judged by the systematic evaluation of two additional moments of the distribution (skewness and kurtosis). These higher moments are exactly zero in the case of a normal distribution and thus we expect that the study of their behaviour should yield sufficient information to influence the decision on the type of approximation required. An improvement to this model consists in the development of the distribution into a serie of orthogonal polynomials. Results using up to sixth-order Hermitian polynomials have been published recently[5].

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3. Simulation results and diagnostics

In this section we compare the simulation results obtained with the delta-function formalism with those of the simplified model described above. All the tables of wake-potentials were obtained with the code BCI[6]. The comparison applies to LEP at injection energy (20GeV) with the bunch-current as independent variable.

Nevertheless, in order to demonstrate the influence of other parameters (such as the momentum compaction α and the related number of synchrotron oscillations per turn ν_s) on the validity of the model, we compare the results for two LEP-lattices (60^0 and 90^0 phase advance). After each turn we evaluate the skewness S and the kurtosis K according to the definitions:

$$S = \frac{1}{N\sigma^3} \sum_{n=1}^{N} (x_n - \bar{x})^3 \tag{5}$$

$$K = \frac{1}{N\sigma^4} \sum_{n=1}^{N} (x_n - \bar{x})^4 - 3 \tag{6}$$

3.1 LEP - 60^0 lattice

As can be seen from Table I, for this lattice both formalisms yield very similar bunch lengthening factors in the whole range of currents considered. At the same time we observe that the magnitude of S and K increase with current and tangible deviations from a gaussian distribution occur at currents higher than 0.5mA (see Table I).

Despite the similarity of the global bunch-lengthening, however, we observe that the strongly non-gaussian shapes occuring at the maximum current tested of 1.2 mA artificially introduce increased fluctuations of the centre of the bunch. This phenomenon is clearly pointed out by comparing the following procedures for the evaluation of an averaged r.m.s. bunchlength over M turns (N particles):

$$\bar{\sigma} = \frac{1}{M} \sum_{m=1}^{M} \sigma_m \tag{7}$$

$$\bar{\bar{\sigma}} = \left[\frac{1}{M} \sum_{m=1}^{M} \frac{1}{N} \sum_{n=1}^{N} (x_{mn} - \bar{x})^2 \right]^{1/2} \tag{8}$$

By doing this, we note that these two values are almost identical with the delta-wake formalism, whereas the gaussian model yields a relative difference of over 20%.

Table I

LEP 60^0lattice : Comparison of the bunch length versus single bunch current. (G = Gauss, δ = delta-formalism)

i_0(mA)	S^G	K^G	σ_t^G(ps)	σ_t^δ(ps)
0.3	<0.5	<1.0	41	43
0.6	<1.0	<3.5	47	46
1.2	<2.5	<19	63	65

For comparison: at 1.2 mA S^δ<0.6 and K^δ<3.0

3. 2. LEP - 90^0 lattice

The main difference of this lattice compared to the previous one is a reduction of α by almost a factor two. Already at 0.6 mA the simplified model underestimates the bunchlengthening by 8%. By further increasing the current, we observe the sporadic appearance of negative values of the skewness. Apart from this new feature, the answers obtained up to 1.0 mA - although they underestimate the bunchlengthening - do not give any indication of an incorrect behaviour of the bunch. At 1.2 mA, however, the simplified simulation yields a totally unphysical behaviour of the bunchlength, consisting in the repetitive blow-up shown in the Fig.1.

Looking at the turn-by-turn fluctuations of S and K, we can observe a close relation between the behaviour of the skewness and the blow-up of the bunch. As can be seen from Fig. 2, the skewness firstly exhibits some negative values and then begins to alternate regularily between positive and negative values with increasing magnitude. The change of sign occurs about every 10 turns corresponding approximately to half the synchroton period, and the onset of this periodic mechanism coincides exactly with the blow-up of the bunch. As illustrated in Fig. 2, the subsequent decrease of the bunchlength corresponds to a similar decrease of the magnitude of the oscillations of S. Furthermore, once the bunch has almost reached its original dimensions (2000 turns) the skewness returns to small and positive values. The next appearance of negative values coincide with a new blow-up. Figs.2 and 3 illustrate the discrepancies between the two models by comparing the values of the corresponding answers of S and K, while Fig. 4 represents the behaviour of the bunch obtained with the delta-function wake formalism for the same conditions.

From these simulations, we observe that the sporadic appearance of negative values of S yields an underestimation of the bunchlengthening, while the onset of the periodic alternation of positive and negative values corresponds to a blow-up situation which precludes any information on the expected bunchlength.

4. Conclusions

The aim of this work on longitudinal simulation of particle stability was to minimize the computing time without abandoning accuracy. This can be achieved by using pre-computed tables of wake-potentials for suitable approximations of the particle distribution. It seemed worthwhile to first test the simplest model which therefore requires the least computing time, i.e. the assumption that the bunch shape remains gaussian.

This study enables us to point out some general features which should be considered in testing higher order approximations. Firstly the skewness and kurtosis show systematically greater deviations from zero than the corresponding results of the delta-function wake formalism. Secondly, the number of synchrotron oscillations per turn ν_s appears to be a crucial parameter for the limitations of the model. Finally - and this seems to be the most powerful tool in such a comparison - the appearance of negative values of the skewness (inexistent in the delta-function formalism) may be interpreted as a warning signal indicating the failure of the model. Considering these arguments, the use of the lowest approximation cannot be regarded as sufficient, as it is not possible to predict the validity of the results with certainty. Since the wake-field part of the simulation is quite sensitive to the asymetry (skewness) of the bunch, this parameter has to be taken into account.

On the other hand, even for strong deviations of the kurtosis, there seems to be little influence on the results. For these reasons we expect a model approximating the bunch distribution as a function of both its r.m.s. standard deviation and its asymetry to be sufficient for most longitudinal tracking simulations. Such a model has the advantages of being faster than the delta-function approach and at the same time requiring less space allocation than an expansion of the particle distribution into many orthogonal polynomials.

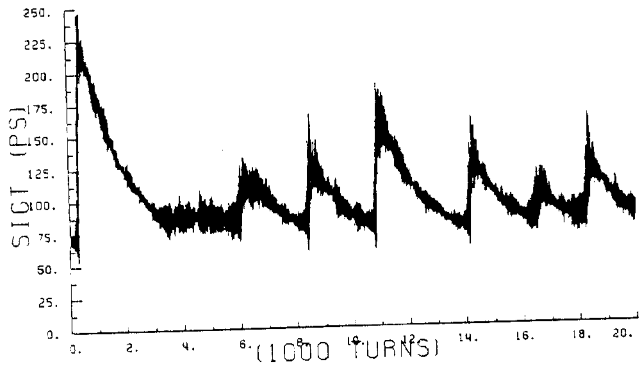

Fig.1 Bunchlength versus number of turns (Gauss)

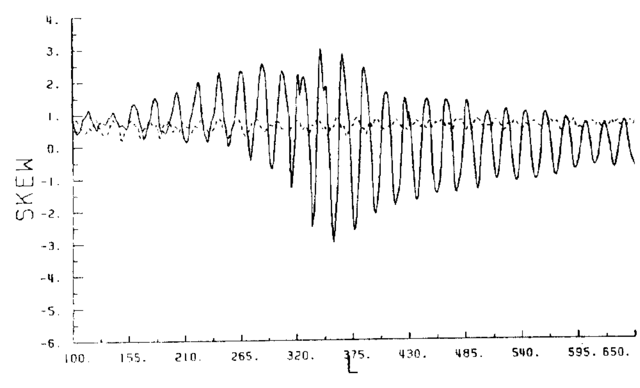

Fig.2 Comparison of the skewness (full line = Gauss; dotted line = δ-wake)

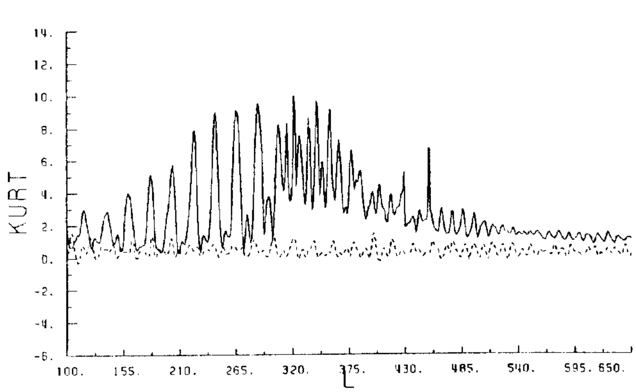

Fig.3 Comparison of the kurtosis (full line = Gauss; dotted line = δ-wake)

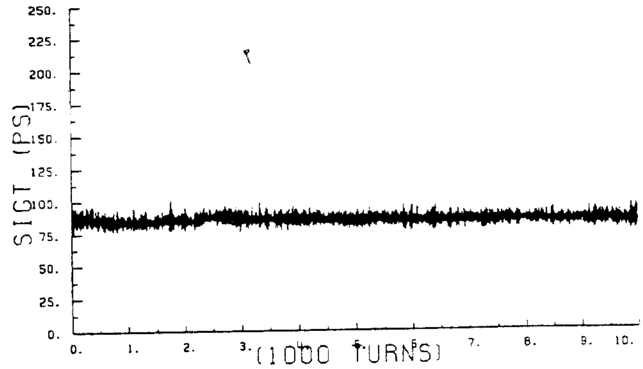

Fig.4 Bunchlength versus number of turns (δ-wake formalism)

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