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A COUPLED EIGENMODE THEORY APPLIED TO RF STRUCTURES

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Abstract

The purpose of the paper is to present a comprehensive analysis of the coupling mechanism between waves propagating in a periodic system of complex structure. The theory is developed in such a way as to utilize results from computer programs, based upon the finite-element method, to solve the two-dimensional eigenvalue problem. We first study a single system, simple enough to be considered as being characterized by two regions, for which the eigenmodes are all known. A matrix analysis is next developed and applied to an n-port circuit, representing the whole system considered as an assembly of subsystems such as studied first. An example is given for illustration. Theoretical results and measurements are compared.

Introduction

It has now been demonstrated that the use of finite element [1,2,3] to solve Maxwell's equations instead of finite difference could enhance the gain in computer time and memory by a factor of 4 to 6. This significant improvement may hasten direct solution of three-dimensional RF problems. In any event, the determination of eigen-solutions for the two-dimensional domain with complex boundaries is no longer an obstacle to an eigenmode-based analysis, by which, under certain conditions, RF structures without cylindrical symmetry could be treated.

We will first consider a single system. Field solutions are written in DINI series and the dispersion relation is obtained by field matching. This technique was applied by Voisin [4] to simple shapes for which eigenfunctions are known analytically. The method is generalized here.

A dispersion relation is obtained in a closed form allowing easy interpretation and approximation. An extension to more complex systems will be carried next, using an n-port circuit representation.

Two-Segment Single Systems

Let us consider the two-segment system of Figure 1. Each segment is characterized by a section, S_1 or S_2, surrounded by a boundary, C_1 or C_2, for which are assumed to be known the TE and TM-eigenfunctions Ψ_m and φ_m, defined by :

$$(\Delta t + \mu_m^2)\ \ \Psi_m = 0, \quad (d\Psi_m/d\vec{n})_c = 0 \tag{1}$$

$$(\Delta t + \nu_m^2)\ \ \varphi_m = 0, \quad (\varphi_m)_c = 0 \tag{2}$$

where Ψ and φ are orthonormalized and depend only on the transverse coordinate, suffix t designating derivation with respect to this coordinate. The conventional field-expansion technique consists in writing the TE and TM-Hertz vectors, which are reduced in our case to their z-component, as a product of the transverse eigenfunctions Ψ_m or φ_m by the longitudinal eigenfunctions exp ($\pm j\alpha_m z$) or exp ($\pm j\beta_m z$), where propagation constants α_m or β_m are related to eigenvalues μ^2 or ν^2 and frequency ω by :

$$\mu_m^2 = (\frac{\omega}{c})^2 - \alpha_m^2\ ,\ \nu_m^2 = (\frac{\omega}{c})^2 - \beta_m^2\ . \tag{3}$$

Alternatively, trigonometric functions can be used instead of exponentials. Then the longitudinal components of the magnetic and electric fields can be written as :

$$H_z = \sum_n \left[X_n\ F_2\ (\alpha_n z) - j\ Y_n\ F_1\ (\alpha_n z)\right] \Psi_n\ (r) \tag{4}$$

$$E_z = \sum_n \left[W_n\ F_2\ (\beta_n z) - j\ S_n\ F_1\ (\beta_n z)\right] \varphi_n (r) \tag{5}$$

The z-dependent odd and even functions F_1 and F_2 are chosen such that they take values of $\pm$ 1 at boundaries z = $\pm$ a, a, standing for a_1 or a_2 depending on the segment considered :

$$F_1\ (\alpha_m z) = \frac{\sin \alpha_m z}{\sin \alpha_m a},\ F_2\ (\alpha_m z) = \frac{\cos \alpha_m z}{\cos \alpha_m a} \tag{6}$$

X_n and Y_n, the even and odd components of the TE mode, and W_n and S_n, the corresponding components of the TM mode, can be taken as components of a complex current vector $I_n(X_n, Y_n)$ and a complex voltage vector $V_n(W_n, S_n)$, representing respectively the two modes. From (4) and (5), transverse fields can be derived. At the boundaries $\pm$ a, they can be written :

$$E\ (a) = \sum_n \left[\frac{\omega\mu o}{\mu_n{}^2}\ R\ I_n\ \vec{k}\ x \nabla\Psi_n - \frac{1}{\nu_n{}^2}\ T(\beta_n a) R\ V_n \nabla\varphi_n\right], \tag{7}$$

$$E\ (-a) = -P\ E(a), \tag{8}$$

$$H\ (a) = \sum_n \left[-\ \frac{1}{\mu_n{}^2}\ T(\alpha_n a)\ I_n \nabla\Psi_n - \frac{\omega\epsilon o}{\nu_n{}^2}\ V_n\ \vec{k}\ x \nabla\varphi_n\right], \tag{9}$$

$$H\ (-a) = -P\ H(a)\ , \tag{10}$$

where suffix t is omitted for simplicity from the field components and from the transverse derivation. R, P, and T are defined as :

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ ,\ P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ ,\ T(\alpha_n a) = \alpha_n \begin{pmatrix} tg\alpha_n a & 0 \\ 0 & 1/tg\alpha_n a \end{pmatrix}. \tag{11}$$

With fields written in their new forms of (7) and (9), a translation of L is represented by ϕ defined as :

$$\phi = \begin{pmatrix} \cos\beta L & -\sin\beta L \\ \sin\beta L & \cos\beta L \end{pmatrix}. \tag{12}$$

If field expressions in each segment are written in their own coordinate system centered on O_1 and O_2, one can write the continuity of transverse fields at boundaries z = $\pm\ a_1$ and z = $\pm\ a_2$ as :

$$E_1\ (a_1) = \phi^{-1}\ E_2\ (-a_2), \tag{13}$$

$$E_1\ (-a_1) = \phi^2\ \phi^{-1}\ E_2\ (a_2), \tag{14}$$

where ϕ is the half-cell matrix. Using Eq(8) and (10) and observing that ϕ^{-1} P = Pϕ, Eq(13) and (14) are reduced to a single equation, yielding :

$$f(E) \equiv E_1\ (a_1) + \phi^{-1}\ PE_2\ (a_2) = 0\ , \tag{15}$$

$$f(H) \equiv H_1\ (a_1) + \phi^{-1}\ PH_2\ (a_2) = 0. \tag{16}$$

Using the orthogonality property of functions Ψ_m an φ_m, Eq(15) and (16) can be written in the following integral forms :

$$\int_{S_2} \vec{k}\ x \nabla\Psi_{2_n} \cdot f(E)\ dS = 0, \tag{17}$$

$$\int_{S_2} \nabla\varphi_{2_n} \cdot f(E)\ dS = 0, \tag{18}$$

$$\int_{S_1} \vec{k}\ x \nabla\varphi_{1_n} \cdot f(H)\ dS = 0, \tag{19}$$

$$\int_{S_1} \nabla\Psi_{1_n} \cdot f(H)\ dS = 0, \text{ for all n}\ . \tag{20}$$

Integration of Eq(17) - (20) is performed by introducing the coupling coefficients :

$$h_{mn} = \int_{S_1} \Psi_{2m}\Psi_{1n} dS, \qquad e_{mn} = \int_{S_1} \varphi_{2m}\varphi_{1n} dS,$$
$$eh_{mn} = \int_C \varphi_{2m} \frac{d\Psi_{1n}}{dc}\ , \tag{21}$$

wich finally yields the system :

$$\begin{aligned} &\phi^{-1} J_2 = h\ P\ J_1\ , \\ &\phi\ \ V_1 = -\tilde{e}\ P\ V_2\ , \\ &\phi\ \ T_1 J_1 = -\ \mu_1^2\ \tilde{h}\ \frac{1}{\mu_2^2}\ T_2\ P\ J_2 - (\epsilon o\omega)\widetilde{eh}\ \frac{1}{\nu_2^2}\ P\ V_2\ , \\ &\phi^{-1} T_2'\ V_2 = (\omega\mu_0)\ eh\ \frac{1}{\mu_1^2}\ P\ J_1 + \nu_2^2\ e\ \frac{1}{\nu_1^2}\ T_1' P V_1\ , \end{aligned} \tag{22}$$

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where h, e, eh and their transposed matrix have elements defined by (21) ; μ^2, $1/\mu^2$ ν^2, $1/\nu^2$ are diagonal matrixes of elements ($\mu_{11}, \mu_{12}, \ldots \mu_{1n}$) etc. , the first suffix being the segment suffix. T and T′ are diagonal matrixes of elements T_{in}, T'_{in}, i = 1, 2, with :

$$T_{in} = \alpha_{in} \begin{pmatrix} tg\alpha_{in}ai & 0 \\ 0 & 1/tg\alpha_{in}ai \end{pmatrix}, \quad T'_{in} = \beta_{in} \begin{pmatrix} 1/tg\beta_{in}ai & 0 \\ 0 & tg\beta_{in}ai \end{pmatrix}. \tag{23}$$

The dispersion relation is obtained by cancelling the determinant of (22). However, the simple form of the first two equations allows elimination of two out of the four vectors. The choice depends on the coupling configuration of interest. If, for instance, the passband that results from coupling between the TE modes of segment 1 and the TM modes of segments 2 is to be made evident, (J_1, V_2) has to be conserved and Eq(22) is reduced to :

$$\begin{pmatrix} \phi T_1 \phi + \mu_1^2 \, \tilde{h} \, \frac{T_2}{\mu_2^2} h & -(\omega\epsilon o) \, \tilde{eh} \, \frac{1}{\nu_2^2} \, \phi \\ (\omega\mu o) \, eh \, \frac{1}{\mu_1^2} & \phi^{-1} T'_2 \phi^{-1} + \nu_2^2 \, e \, \frac{T'_1}{\nu_1^2} \, \tilde{e} \end{pmatrix} \begin{pmatrix} J_1 \\ -\phi^{-1} P V_2 \end{pmatrix} = 0 \tag{24}$$

The diagonal terms can be interpreted as the dispersion relation of the TE and TM modes when cross-coupling is neglected, and the others as resulting from this coupling. As every term is Eq(24) is a diagonal matrix except where ϕ appears, the effect of ϕ is to mix the even and odd modes together. If ϕ is itself a diagonal matrix, i.e., at zero or π -mode, all terms in Eq(24) are diagonal, then Eq(24) is split into two independent systems corresponding to pure odd or even solutions. The present case can be easily extended to a four-segment structure with interdigital symmetry. That is the case of interdigital structures themselves and also of a cavity chain where coupling holes are alternated with 180° rotation. Eq(24) then has to be modified slightly by replacing T_2 and T'_2 by the opposite of their symmetric.

As a finite-element method provides the possibility of solving the eigenvalue problem with the least effort, Eq(24) gives a simple way to obtain the dispersion curves and field components. A vane structure, with the dimensions given in Figure 2, is taken as an example. The coupling coefficients of the first six modes are calculated from Eqs(21) and tabulated in Table 1. Their distribution is instructive : the separation between symmetrical (odd suffix) and antisymmetrical (even suffix) coefficients is clearly seen, as well as the importance of coupling between modes. Figure 2 shows that, for the first passband, even with only four modes, the agreement between the theoretical curve and the measured values is already excellent.

TABLE 1

Segment 2 / 1	TE01 4.28(*)	TE11 10.87	TE03 12.85	TM11 10.87	TE02 8.57	TM12 13.16
TE01 3.61	0.988	0.042	0.025	0.673	0	0
TE11 10.55	0.070	0.971	0.089	0.829	0	0
TE03 12.62	0.057	0.138	0.946	1.038	0	0
TM11 13.82	0.673	0.829	1.038	0.840	0	0
TE02 8.46	0	0	0	0	0.835	0.788
TM12 14.40	0	0	0	0	0.788	0.950

(*) Frequency in GHz

Circuits of Complex Structure

In electron devices, sometimes more than one circuit are necessary to achieve the desired dispersion characteristics and more than one passband have to be considered in order to prevent spurous modes. The theory developed above could be a useful tool to analyze the behavior of such a system. First, neglecting coupling between circuits, equations similar to Eq(24) are obtained for each of them. Next, by introducing coupling matrixes, defined similarly by matching tranverse fields on coupling areas, equations for the whole system can be derived and solved. This method will be developed here only in a simpler form by using lumped element circuits.

Let us consider a case where all the circuits are coupled through a single element, which is a resonant slit in the example considered here. Coupling is represented by a coupling cell made up of n such circuits, as shown in Figure 3, connected to the same impedance Z representing the coupling element. Each circuit is constituted of a parallel impedance zi, a mesh impedance Zi representing the circuit AiBiCiDiAi, and two current-divider impedances ξi and ηi. The latter are determined so that the ratio $\alpha i = \xi i/(\xi i + \eta i)$ corresponds to the contribution of the i-th mode current to the excitation of Z and so that, in the absence of Z, the original uncoupled circuit is reconstituted.

By considering the voltage-current vector (V,I) of n-elements [(V_1, I_1), . . . (V_n, I_n)], the transfer matrix of the coupling cell can be obtained. Its elements are written as :

$$Mii = \begin{pmatrix} -1 + mii & zi\,(2 - mii) \\ -\frac{mii}{zi} & -1 + mii \end{pmatrix}, \tag{25}$$

and

$$Mij = \frac{mij}{zj} \begin{pmatrix} zi & -zizj \\ -1 & zj \end{pmatrix}, \tag{26}$$

where

$$mii = \frac{Zi}{zi} - \frac{\alpha i \xi i}{zi} \left[1 - \frac{\alpha i}{\xi i} \left(\frac{1}{z} + \frac{\alpha_1}{\xi_1} + \ldots + \frac{\alpha_n}{\xi_n} \right)^{-1} \right], \tag{27}$$

and

$$mij = \frac{\alpha i \alpha j}{zi} \left(\frac{1}{z} + \frac{\alpha_1}{\xi_1} + \ldots + \frac{\alpha_n}{\xi_n} \right)^{-1}. \tag{28}$$

The circuit for one period is constructed around this coupling cell, and we assume that its transfer matrix M is known.

The main problem we have to deal with is finding the eigenvalues of the transfer matrix. If we introduce the simplectic conjugate matrix defined as $\bar{M} = -S\,\tilde{M}\,S$ where S is a diagonal (n x n) matrix of element $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and if λ is an eigenvalue, then λ^{-1} is an eigenvalue of $\bar{M}$, as a consequence of the reciprocity property. If λ is now written in the form $\lambda = \exp(j\phi)$, then the dispersion relation can be written as :

$$F(\omega) = \cos\phi(\omega) = \frac{1}{2}\,(\lambda + \lambda^{-1}), \tag{29}$$

which is none other than one-half of the eigenvalue of the sum :

$$A = M + \bar{M}. \tag{30}$$

This yields the characteristic equation :

$$A - 2\,F(\omega)\,.\,I = 0, \tag{31}$$

where I is an (nxn) unit matrix.

As an example, let us consider the periodic structure shown in Figure 4, commonly used in coaxial magnetrons and which could be a candidate for a millimetric-wave device. The structure is made up of a vane circuit used as an interaction circuit and a rectangular waveguide, by which RF power is removed. The two circuits are coupled together through a biperiodic row of slits. It is known that in vane structures the π mode is the most effective, but is impracticable if vanes alone are employed because of the poor mode separation. Through the biperiodic slits, this mode is coupled to the TM zero mode of the wavegguide, which behaves as a mode filter in imposing its mode separation. Unfortunately, the complexity of the structure gives rise to other unwanted modes, which have to be selectively damped and which we propose to study here.

In order to give an acceptable description of the first passbands, at least four modes have to be taken into account, namely, the vane mode, the waveguide TE and TM modes and the lowest TE modes of the slits, resulting in a six-port circuit from which an A-matrix can be derived. The dispersion relation F is obtained by solving a three-degree characteristic equation (31). Details of how to obtain all the lumped impedances of the model are not given here. It can be said, however, that the knowledge of the zero and π modes is sufficient to derive the π-shaped equivalent circuit of Figure 3.

Figure 5 shows theoretical and experimental results. One can see that the dispersion curves of the uncoupled system, shown by dashed lines, are deeply affected by the presence of slits. Starting from the low-frequency end, the first branch comes from the lower half of the vane initial curve divided by the slit resonance. The second and third are generated from coupling between the waveguide TE and vane modes. The second half of the vane curve disappears and the previous vane π mode then belongs to the waveguide TM branch, as expected. Finally, coupling between waveguide TE and TM modes is observed at the high frequency end of the Figure. The theory thus predicts the experimental results quite well.

Conclusion

A method is proposed to derive field and dispersion curve for periodic RF circuit of complex structure. With the theory developed in the first part, some useful circuits, for example vane, interdigital structure, cavity chain or iris-load structure with thick and non circular iris, could be treated without need of a three dimension program. In the second part, assembly of subsystems is considered. An example is given showing the possibility of the method to analyze complex system.

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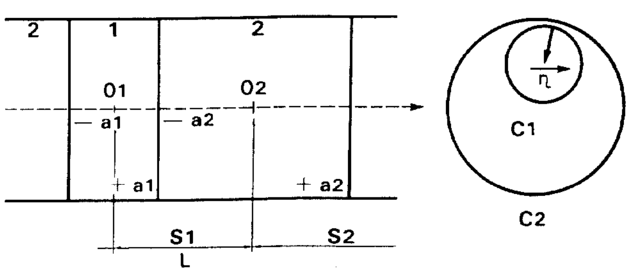

Figure 1 - General view of a two-segment structure

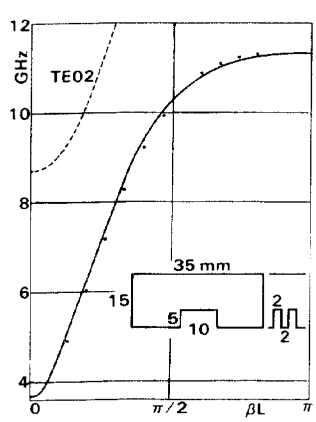

Figure 2 - Dispersion curves for a vane structure

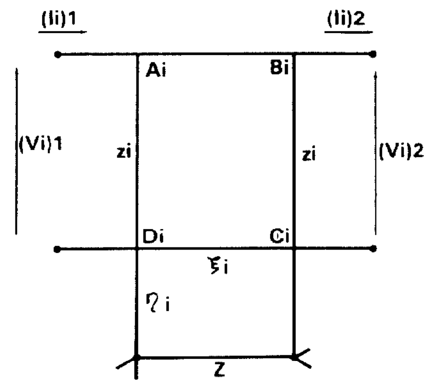

Figure 3 - Equivalent circuit

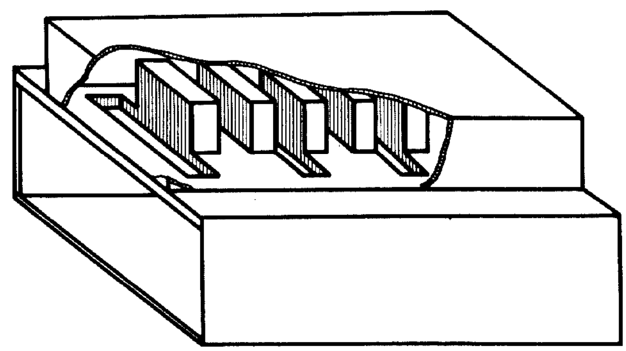

Figure 4 - Vane structure periodically coupled to a waveguide

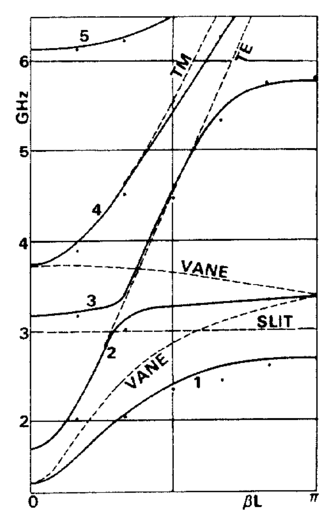

Figure 5 - Dispersion curves for a vane structure biperiodically couped to a waveguide