THE HIGHER ORDER MODES CALCULATION OF RF CAVITY WITH CYLINDRICAL SYMMETRY

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Abstract

This paper represents a new computer calculating method for cylindrical symmetry cavities. This method can calculate not only the fundamental mode and longitudinal modes but also the transverse higher modes.

The Hertz vector is used as fundamental quantity and the separated variable method is applied.

An empty cylindrical cavity has been calculated. The calculating result is in accord with analytic solution fairly well.

Introduction

The reentrant cylindrical cavities is used in high power klystron, some solid-state devices and electron storage ring. The calculating method of this kind of cavities has been introduced by many papers.[1-6]

In recent years, the computer code SUPERFISH is widely used for axisymmetric cavities of arbitrary shape, which is very succesful, but the SUPERFISH can calculate the axisymmetric rf field only, i.e. for $\partial E/\partial\phi = 0, \partial H/\partial\phi = 0$ condition. On principle, it can evaluate longitudinal higher modes and cannot evaluate transverse higher modes. Of course, we can use finite element method to calculate three dimensional field, but the capacity of computer required will be too large and the expenditure of computer time is too great. It is not suitable for engineering use.

Principle and Fundamental Equations

In SUPERFISH and other calculating methods, in general, $\partial E/\partial\phi = 0$, $\partial H/\partial\phi = 0$, i.e. for axisymmetric fields was assumed. In this condition, along ϕ direction fields are not varied, so the three dimensional problem can easily represented by two dimensional fields and two independent sets of solutions can exist: TE mode having three nonzero field components E_ϕ, H_r, H_z, and the TM mode H_ϕ, E_r, E_z. In each set, the fundamental quantity H or E only have one angular component. So it is very convenient to choose $H = H_\phi$, $E = E_\phi$ as fundamental quantity to calculate the TM and TE modes.

If $\partial H/\partial\phi, \partial E/\partial\phi \neq 0$, fields are not axisymmetric, and with angular variation. There are five fundamental variable quantities in TM and TE modes. H and E no longer have one angular component. In this case, it isn't convenient to choose H_ϕ or E_ϕ as fundamental quantity. However, the variation of fields along angular direction is periodic, the variation of phase is $2n\pi$ a turn, n is an integer, representing the variation period, n = 0 represents no variation along angular direction. So the Separated Variable Method can be applied to separate the angular quantity thus simplify the three dimensional problem to two dimensional problem yet. In addition, if we use the Hertz electric vector and Hertz magnetic vector as fundamental quantities, we can find the five fundamental quantities easily. So this method may be called "The Separated Variable Hertz Vector Method".

For TM modes, the Hertz electric vector can be used. In cavity space, the Hertz electric vector wave equation is:

$$\nabla^2\vec{\Pi}_e + K^2\vec{\Pi}_e = 0 \tag{1}$$

where $K^2 = \omega^2\varepsilon\mu$

and $$\vec{E} = \frac{1}{\varepsilon}(\text{grad div}\vec{\Pi}_e + K^2\vec{\Pi}_e)e^{j\omega t} \tag{2}$$

$$\vec{H} = j\omega\,\text{rot}\,\vec{\Pi}_e e^{j\omega t} \tag{3}$$

For TM modes, the Hertz electric vector lies along z direction

$$\begin{aligned}\Pi_e &= \Pi_{ez}\\ \Pi_{e\phi} &= \Pi_{er} = 0\end{aligned} \tag{4}$$

Thus the $H_z = 0$ condition is satisfied.

Then

$$\begin{aligned}
E_r &= \frac{1}{\varepsilon}\frac{\partial^2\Pi_e}{\partial r\partial z}e^{j\omega t} & H_r &= j\omega\frac{1}{r}\frac{\partial\Pi_e}{\partial\phi}e^{j\omega t}\\
E_\phi &= \frac{1}{\varepsilon\gamma}\frac{\partial^2\Pi_e}{\partial\phi\partial z}e^{j\omega t} & H_\phi &= -j\omega\frac{\partial\Pi_e}{\partial r}e^{j\omega t}\\
E_z &= \frac{1}{\varepsilon}\left(\frac{\partial^2\Pi_e}{\partial z^2} + K^2\Pi_e\right)e^{j\omega t} & H_z &= 0
\end{aligned} \tag{5}$$

Like this, for TE modes, the Hertz magnetic vector wave equation can be used.

$$\nabla^2\vec{\Pi}_m + K^2\vec{\Pi}_m = 0 \tag{6}$$

$$\vec{E} = j\omega\,\text{rot}\,\vec{\Pi}_m e^{j\omega t} \tag{7}$$

$$\vec{H} = -\frac{1}{\mu}(\text{grad div }\vec{\Pi}_m + K^2\vec{\Pi}_m)e^{j\omega t} \tag{8}$$

For TE modes, assuming Hertz magnetic vector lies along z direction i.e.

$$\begin{aligned}\Pi_m &= \Pi_{mz}\\ \Pi_{mr} &= \Pi_{m\phi} = 0\end{aligned} \tag{9}$$

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On this condition, $E_z = 0$ is satisfied.

$$E_r = j\omega \frac{1}{r}\frac{\partial \Pi_m}{\partial \phi} e^{j\omega t} \qquad H_r = -\frac{1}{\mu}\frac{\partial^2 \Pi_m}{\partial z \partial r} e^{j\omega t}$$

$$E_\phi = -j\omega \frac{\partial \Pi_m}{\partial r} e^{j\omega t} \qquad H_\phi = -\frac{1}{\mu}\frac{1}{r}\frac{\partial^2 \Pi_m}{\partial \phi \partial z} e^{j\omega t} \tag{10}$$

$$E_z = 0 \qquad H_z = -\frac{1}{\mu}\left(\frac{\partial^2 \Pi_m}{\partial z^2} + K^2 \Pi_m\right)e^{j\omega t}$$

Hertz vector $\vec{\Pi}$ is a function of r, z, and ϕ three variables, applying Separated Variable Method, let

$$\Pi = \Phi U \tag{11}$$

U is a function of r and z.
Φ is a function of ϕ.
Insert (11) into wave equation with cylindrical coordinate, we can obtain two formulas.

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} + \left(K^2 - \frac{n^2}{r^2}\right) U = 0 \tag{12}$$

$$\frac{\partial^2 \Phi}{\partial \phi^2} + n^2 \Phi = 0 \tag{13}$$

The solution of equation (13) is

$$\Phi = A \cos(n\phi + \phi_n)$$

Choosing the reference plane appropriately, we can make $\phi_n = 0$, if we consider field distribution only, we can put A = 1, then

$$\Pi = U \cos n\phi \tag{14}$$

For TM modes

$$E_r = E_{rm} \cos n\phi \qquad H_r = H_{rm} \sin n\phi$$

$$E_\phi = E_{\phi m} \sin n\phi \qquad H_\phi = -H_{\phi m} \cos n\phi \tag{15}$$

$$E_z = E_{zm} \cos n\phi \qquad H_z = 0$$

and

$$E_{rm} = \frac{1}{\varepsilon}\frac{\partial^2 U_e}{\partial r \partial z} \qquad H_{rm} = -j\omega \frac{n}{r} U_e$$

$$E_{\phi m} = -\frac{n}{\varepsilon r}\frac{\partial U_e}{\partial z} \qquad H_{\phi m} = -j\omega \frac{\partial U_e}{\partial r} \tag{16}$$

$$E_{zm} = \frac{1}{\varepsilon}\left(\frac{\partial^2 U_e}{\partial z^2} + K^2 U_e\right)$$

For TE modes

$$E_r = E_{rm} \sin n\phi \qquad H_r = H_{rm} \cos n\phi$$

$$E_\phi = E_{\phi m} \cos n\phi \qquad H_\phi = H_{\phi m} \sin n\phi \tag{17}$$

$$E_z = 0 \qquad H_z = H_{zm} \cos n\phi$$

and

$$E_{rm} = -j\omega \frac{n}{r} U_m \qquad H_{rm} = -\frac{1}{\mu}\frac{\partial^2 U_m}{\partial z \partial r}$$

$$E_{\phi m} = -j\omega \frac{\partial U_m}{\partial r} \qquad H_{\phi m} = \frac{1}{\mu}\frac{n}{r}\frac{\partial U_m}{\partial z} \tag{18}$$

$$H_{zm} = -\frac{1}{\mu}\left(\frac{\partial^2 U_m}{\partial z^2} + K^2 U_m\right)$$

It is convenient obviously to apply separated variable Hertz vector method and choose U as the fundamental quantity. The crux of the problem is the treatment of boundary condition.

The cylindrical double reentrant axisymmetric cavities is symmetry from left to right and from up to down. So calculating 1/4 section is enough.

For z = 0 axis, the field distribution can exist in two possible forms, odd symmetry or even symmetry. At z = 0 plane the boundary condition is

$$\begin{aligned} &U = 0 \text{ (odd symmetry)} \\ &\partial U/\partial z = 0 \text{ (even symmetry)} \end{aligned} \tag{19}$$

At r = 0 axis, if n = 0, based on the symmetry condition of up to down, so

$$\partial U/\partial r = 0 \tag{20}$$

If $n \neq 0$ from (12), because the field value at r = 0 axis is definite, at r = 0 axis

$$U = 0$$

From the basic physical properties of metallic surface, the boundary condition is

$$\vec{n} \times \vec{E} = 0 \tag{21}$$

$$\vec{n} \cdot \vec{H} = 0 \tag{22}$$

Similarly,

$$\vec{n} \times \vec{\Pi}_e = 0 \tag{23}$$

$$\vec{n} \cdot \vec{\Pi}_m = 0 \tag{24}$$

$\vec{n}$ represents the unit normal vector on the surface. i.e. the Hertz electric vector and electric field is perpendicular to metallic surface and the Hertz magnetic vector is parallel to the metallic surface.

For the TM modes, the $\vec{\Pi}_e$ is parallel to z axis. On the other hand, $\vec{\Pi}_e$ is perpendicular to metallic surface. If the metallic surface is not perpendicular to z axis, the two conditions are mutually exclusive thus U must be equal to zero at this surface. For metallic surface perpendicular to z axis, according to $E_{rm} = E_{\phi m} = 0$, from equation (16) $\partial U/\partial z = 0$.

For the TE modes, $\vec{\Pi}_m$ is parallel to z axis. If the metallic surface is not parallel to z axis, the condition of $\vec{\Pi}_m$ parallel to z axis and metallic surface are mutally exclusive, thus U must be equal to zero at this metallic surface. For the metallic surface parallel to z axis, according to $E_{\phi m} = H_{rm} = 0$, from equation (18) $\partial U/\partial r = 0$.

Numerical Calculation

According to the principle and fundamental equations mentioned above, we can perform the numerical calculation. In this paper the differential method is accepted. We use the equal mesh, five point differential pattern and iterative method.

Each mode corresponding to a linear equation, all the nodes compose a set of linear equations. To represent as matrix form

$$AU = 0 \tag{25}$$

A is the coefficient matrix. The characters of A are sparse, most of the elements are zero and the elements on diagonal line are superior in numbers, and all the elements on diagonal line $a_{ii} > 0$. It can be turned to a symmetric matrix approximately, so the successive over relaxation method can be used.

The resonant frequency of each mode can be calculated with Rayleigh quotient. Multiply U both side of equation (12) and intergrate in the whole region

For TM Modes:

$$K^2 = \frac{-\int_s U_e\left(\frac{\partial^2 U_e}{\partial r^2} + \frac{1}{r}\frac{\partial U_e}{\partial r} + \frac{\partial^2 U_e}{\partial z^2} - \frac{n^2}{r^2}U_e\right)dS}{\int_s U_e^2 dS} \tag{26}$$

For TE Modes:

$$K^2 = \frac{-\int_s U_m\left(\frac{\partial^2 U_m}{\partial r^2} + \frac{1}{r}\frac{\partial U_m}{\partial r} + \frac{\partial^2 U_m}{\partial z^2} - \frac{n^2}{r^2}U_m\right)dS}{\int_s U_m^2 dS} \tag{27}$$

The resonant frequency $f = \frac{Kc}{2\pi}$

Where c is the phase velocity in vacuum.

The field values can be derived from U_e, U_m with formulas (15) - (18) directly.

We used double iterative method. The initial value of U and K is given by suppositon at beginning, then we calculate the new value of U with Successive Over Relaxation Method, and get the new value of K from Rayleigh quotient. To do this with repeat circulation up to the required accuracy is obtained.

For calculating the eigenvalue with iterative method, the iterative process is always convergent to the 1st eigenvalue. In order to calculate the high order modes, we want to find the high order eignevalues.

Let $V^{(n)}$ be the n dimensions vector space, the fundamental vector is composed of all the eigenvectors in Matrix A.

$$\vec{U} = \sum_{i=1}^{n} c_i \vec{U}_i \tag{29}$$

For the orthogonal character of eigenvectors in $V^{(n)}$ space.

$$c_i = \frac{\vec{U}\cdot\vec{U}_i}{\vec{U}_i\cdot\vec{U}_i} \tag{30}$$

In calculating process, at first the initial vector is given and the 1st eigenvalue $\vec{U}_1$ is found. Then c_1 can be found from (30). At the next step, let $\vec{U}-c_1\vec{U}_1$ be the initial vector and the 2nd eigenvalue $\vec{U}_2$ can be found.

Similarly, let $\vec{U} - c_1\vec{U}_1 - c_2\vec{U}_2$ be the initial vector, $\vec{U}_3$ can be found. Step by step we can find the higher eigenvalue in due order.

The Result of Calulation

For checking this prescribed calculating method, we calculate an empty cylindrical cavity with a relative coarse mesh. The calculating result is in accord with analytic solution fairly well. It proves that this calculating method and formulas mentioned above are corrected and feasible.

Table 1. The Resonant Frequency of Cylindrical Cavity

(D=76 mm; L=68 mm) (mesh size: 2x2 mm)

Mode	Analytic solution (MHz)	Calculating Value (MHz)	Calculating error
TM_{010}	3019.424	3016.7896	8.72×10^{-4}
TM_{012}	5343.1526	5333.6836	1.77×10^{-3}
TM_{020}	6930.237	6966.8643	5.29×10^{-3}
TM_{110}	4810.99	4805.9360	1.05×10^{-3}
TM_{112}	6525.178	6520.0591	7.84×10^{-4}
TM_{120}	8832.28	8690.2617	1.6×10^{-2}
TE_{011}	5291.82	5276.0342	2.98×10^{-3}
TE_{111}	3193.779	3193.4961	8.86×10^{-6}
TE_{211}	4422.57	4421.5918	2.21×10^{-4}
TE_{112}	4891.41	4971.9780	1.89×10^{-3}
TE_{212}	5847.10	5837.4873	1.64×10^{-3}
TE_{012}	6525.154	6527.6670	3.85×10^{-4}
TE_{312}	6879.40	6868.6016	5.05×10^{-3}
TE_{113}	7004.592	7002.7582	2.62×10^{-4}
TE_{313}	8458.08	8433.3652	2.92×10^{-3}

Conclusion

This paper represents a computer calculating method for cylindrical cavities. This method can not only calculate the fundamental mode and longitudinal higher modes but also the transverse higher modes.

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