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SECOND ORDER EFFECTS OF A SEXTUPOLAR FIELD ON BETATRON OSCILLATIONS IN A STORAGE RING†

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Summary

Calculations concerning the betatron oscillations in a storage ring lattice comprising a lumped element sextupole field are presented. The method used is based on the second order approximation of the averaging method. The existence of singular points inside the region of the phase space defined by the separatrix is assumed. Formulas are given permitting the calculations of the size of the corresponding intersecting invariant curves and a simple expression is deduced for the betatron tune shift. Numerical application is made, the results are compared to those obtained by a tracking program. A good agreement between them is noticed.

Introduction

The perturbation effects induced by sextupolar fields on the betatron oscillations in storage rings have been already extensively analysed. Among the features which were pointed out, we will retain those related to second order effects. They are basically of two kinds :

- the dependence of the betatron tune shift on the betatron amplitude and tune,
- the existence of singular points even inside the region of the phase space defined by the separatrix.

Since they appear to be better known through computer simulation analysis, one recollects here some analytical results they are related to.

Equation of Motion

The perturbation induced on betatron motion by n delta function sextupolar fields distributed over the circumference of a storage ring is described by the following system of equations :

$$\frac{dI}{d\theta} = -2\varepsilon\nu I^{3/2}\sin\psi\cos^2\psi \left[\sum_{i=1}^{n}\cos\xi_i\sum_{k=-\infty}^{\infty}\delta(\theta-\theta_i-2\pi k)\right]$$

$$\frac{d\phi}{d\theta} = -\varepsilon\,\nu\, I^{1/2}\cos^3\psi \left[\sum_{i=1}^{n}\cos\xi_i\sum_{k=-\infty}^{\infty}\delta(\theta-\theta_i-2\pi k)\right] \quad (1)$$

derived from the well-known Courant-Snyder equation by the substitution :

$$\eta = \sqrt{I}\cos(\nu\theta+\phi)$$

$$\eta' = -\nu\sqrt{I}\sin(\nu\theta+\phi)$$

$$\psi = \nu\theta+\phi$$

One admits the parameter, bringing out the features of the sextupolar perturbations, expressed by :

$$\varepsilon^2 = \sum_{i=1}^{n}\varepsilon_i^2$$

$$\cos\xi_i = \varepsilon_i/\varepsilon$$

where $\varepsilon_i = \beta^{5/2}(\theta_i)\times k_{sext}$, to be small enough to work out a valuable approximate solution for the considered system of equations with the averaging method. Only second order effects are considered.

Betatron Tune Shift

According to the averaging method, briefly resumed in the Appendix, the following approximate system of equations can be associated to (1) when $m\nu \neq n$:

$$\frac{dI_o}{d\theta} = 0$$

$$\frac{d\phi_o}{d\theta} = -\varepsilon^2\frac{I_o}{32\pi^2}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\cos\xi_i\cos\xi_j\left[\frac{10}{3\nu} + 6\nu\sum_{k=1}^{\infty}\left(\frac{1}{9\nu^2-k^2}+\frac{1}{\nu^2-k^2}\right)\cos k(\theta_i-\theta_j)\right]\right\} .$$

Noticing that[1] :

$$2X\sum_{k=1}^{\infty}\frac{1}{X^2-K^2} = \pi\,\mathrm{ctg}\,\pi X - \frac{1}{X}$$

$$2X\sum_{k=1}^{\infty}\frac{\cos k\theta}{X^2-K^2} = \pi\frac{\cos X(\pi-\theta)}{\sin\pi X} - \frac{1}{X} .$$

One obtains the below expression giving the betatron tune shift caused by n delta function sextupoles distributed on the circumference of a storage ring :

$$\Delta\nu = -\varepsilon^2\frac{I_o}{32\pi}\left\{\sum_{i,j}\cos\xi_i\cos\xi_j\left[\mathrm{ctg}\,3\pi\nu\cos 3\nu(\theta_i-\theta_j) + 3\,\mathrm{ctg}\,\pi\nu\cos\nu(\theta_i-\theta_j)\right]\right\}$$

and, also, the following expressions indicating the conditions for which the induced tune shift vanishes :

$$\sum_{i,j}\cos 3\nu(\theta_i-\theta_j)\cos\xi_i\cos\xi_j = 0$$

$$\sum_{i,j}\cos\nu(\theta_i-\theta_j)\cos\xi_i\cos\xi_j = 0 .$$

For a single delta function sextupole the tune shift is thus given by the following rather simple formula :

$$\Delta\nu = -\varepsilon^2\frac{I_o}{32\pi}(3\,\mathrm{ctg}\,\pi\nu + \mathrm{ctg}\,3\pi\nu) .$$

Results obtained by "turn per turn" simulation and those given by the above formula are gathered in figures 1 to 3. One remarks the consistency between them.

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Motion at the Vicinity of the Singular Point $\nu \approx p/4$

The phase portrait of the betatron motion perturbed by a delta function sextupolar field exhibits a rather complex pattern[2]. Singular points appear inside the region of the phase space defined by the separatrix, described by the expression :

$$3\,\varepsilon\, I_o^{1/2} \cos\psi_o = \varepsilon\Delta$$

$$\psi_o = (\nu - \tfrac{p}{3})\,\theta + \phi_o = \varepsilon\Delta\theta + \phi_o \quad .$$

An analysis, based on the second order approximation of the averaging method, could be carried on for some of them : $\nu \approx p/4$, $\nu \approx p/5$ and $\nu \approx p/6$.

The perturbative effects produced by a delta function sextupolar field for $\nu = p/4 - \varepsilon\Delta$ are considered. They are described by the following system of equations :

$$\frac{dI_o}{d\theta} = \varepsilon^2 \frac{I_o^2}{32\,\pi}(3\,\mathrm{ctg}\,\pi\nu - \mathrm{ctg}\,3\pi\nu)\,\sin 4\psi_o$$

$$\frac{d\psi_o}{d\theta} = \varepsilon\Delta - \varepsilon^2 \frac{I_o}{32\,\pi}\Big[(\mathrm{ctg}\,3\pi\nu + 3\,\mathrm{ctg}\,\pi\nu) + 2\,\mathrm{ctg}\,\pi\nu\,\cos 4\psi_o\Big] \; .$$

For $p = 3$ an invariant relation could be written since one has $3\,\mathrm{ctg}\,\pi\nu - \mathrm{ctg}\,3\pi\nu \approx 4\,\mathrm{ctg}\,\pi\nu$:

$$4\,\varepsilon\Delta\, I_o + \varepsilon^2 I_o^2 \Big[(3\,\mathrm{ctg}\,\pi\nu - \mathrm{ctg}\,3\pi\nu)\cos 4\psi_o + 2(3\,\mathrm{ctg}\,\pi\nu + \mathrm{ctg}\,3\pi\nu)\Big] = \mathrm{Cte} \; .$$

The singular point's amplitudes, calculated from the above relation by putting $dI_o/d\theta = 0$ and $d\phi_o/d\theta = 0$, are plotted on figures 4 and 5 for different values of the dissonance $\varepsilon\Delta$ and sextupole "strength" ε. The dotted curves represent the corresponding relations obtained by "turn per turn" simulation.

References

1 B. Zotter, CERN/ISR-TH/78-9

2 E. Crosbie, T. Khoe, R. Lari, IEEE NS-18 (1971)

3 N. Bogolioubov et N. Mitropolski, Méthodes asymptotiques en théorie des oscillations non-linéaires. Paris, 1962

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Appendix

1) Following the averaging method, it is possible to associate to the system of equations[3] :

$$\frac{dI}{d\theta} = \varepsilon\, f\,(I,\phi,\theta)$$
$$\frac{d\phi}{d\theta} = \varepsilon\, g\,(I,\phi,\theta) \qquad \text{(a-1)}$$

with ε small and

$$f(I,\phi,\theta) = f(I,\phi,\theta + T)$$
$$g(I,\phi,\theta) = g(I,\phi,\theta + T) \; .$$

An approximate system of equations :

$$\frac{dI_o}{d\theta} = \varepsilon\, M\{f(I_o,\phi_o,\theta)\} + \varepsilon^2 M\{\tilde{F}\frac{\partial f}{\partial I_o} + \tilde{G}\frac{\partial f}{\partial \phi_o}\}$$

$$\frac{d\phi_o}{d\theta} = \varepsilon\, M\{g(I_o,\phi_o,\theta)\} + \varepsilon^2 M\{\tilde{F}\frac{\partial g}{\partial I_o} + \tilde{G}\frac{\partial g}{\partial \phi_o}\} \qquad \text{(a-2)}$$

where M, $\tilde{F}(I_o,\phi_o,\theta)$ and $\tilde{G}(I_o,\phi_o,\theta)$ are respectively the averaging operator :

$$M\{f(I_o,\phi_o,\theta)\} = \frac{1}{T}\int_0^T f(I_o,\phi_o,\theta)\, d\theta$$

and

$$\tilde{F}(I_o,\phi_o,\theta) = \int_0^\theta \Big[f(I_o,\phi_o,\theta) - M\{f(I_o,\phi_o,\theta)\}\Big]\, d\theta + \tilde{F}(I_o,\phi_o,0)$$

$$\tilde{G}(I_o,\phi_o,\theta) = \int_0^\theta \Big[g(I_o,\phi_o,\theta) - M\{g(I_o,\phi_o,\theta)\}\Big]\, d\theta + \tilde{G}(I_o,\phi_o,0)$$

from which exact solutions are the approximate one to $\sigma(\varepsilon^3)$ order of the original system of equations (a-1).

2) The turn per turn simulation is done according to the following relations :

$$\eta_{n+1} = \eta_n \cos 2\pi\nu + \delta_n \sin 2\pi\nu$$

$$\eta'_{n+1} = -\eta_n \sin 2\pi\nu + \delta_n \cos 2\pi\nu - \frac{\varepsilon\,\eta_{n+1}^2}{2}$$

with

$$\delta_n = \eta'_n - \frac{\varepsilon\,\eta_n^2}{2} \; .$$

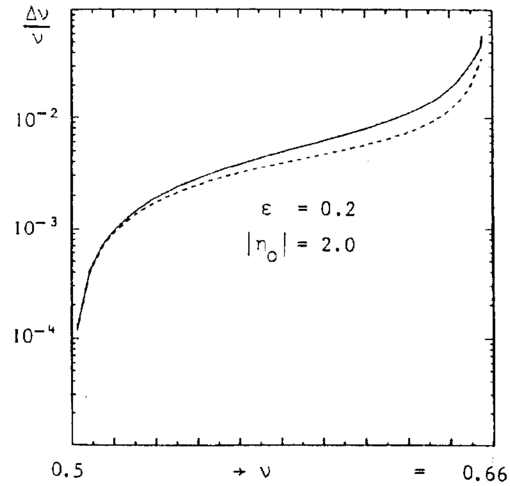

Figure 1

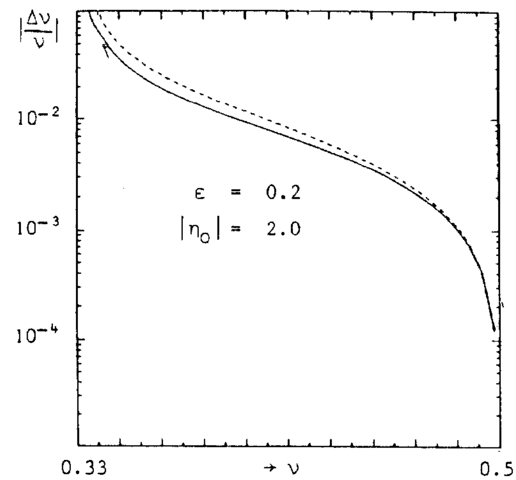

Figure 2

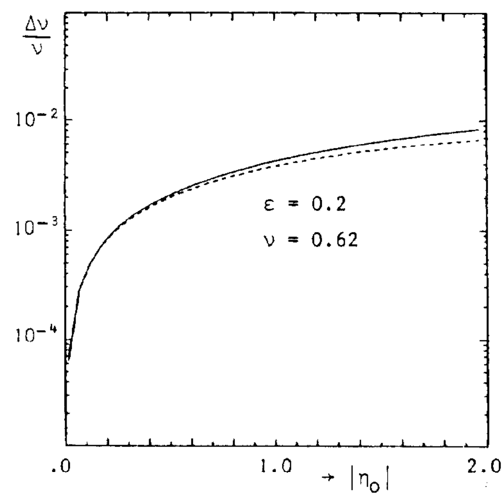

Figure 3

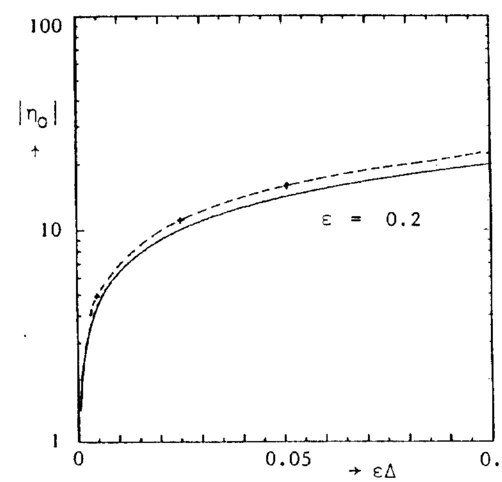

Figure 4

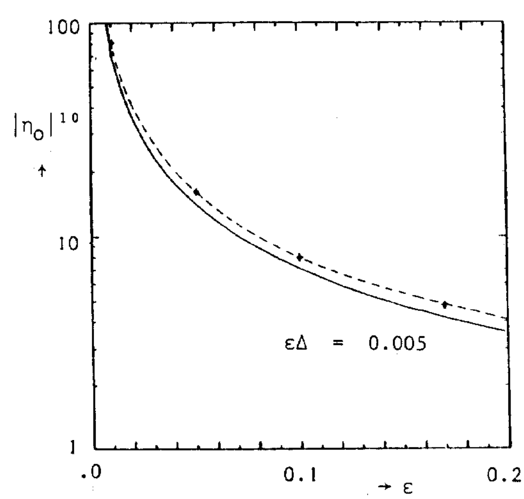

Figure 5