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ANALYTIC ESTIMATES FOR THE DYNAMIC APERTURE OF NON-LINEAR LATTICES

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Summary

For an accelerator lattice containing non-linear elements, particle amplitudes outside a "dynamic aperture" are unstable. In this paper an analytic procedure for estimating this aperture not depending on resonance assumptions is described. The amplitude $x(N)$ after N iterations is expanded in powers of the initial amplitude x_0. Low order coefficients are obtained using the symbolic algebraic manipulation program REDUCE. For intermediate values of N (e.g., 100) these terms give an asymptotic description which, by particle tracking, is shown to be valid. This leads to an estimate of the dynamic aperture and to its possible enlargement.

General Method

The storage ring modelled will consist of long purely linear sections and thin non-linear elements. We will use normalized coordinates (1) so that the linear elements will be totally characterized by the betatron phase advance through them.

$$x = X/\zeta \quad , \quad u = dx/d\phi = U\zeta + \alpha X/\zeta \tag{1}$$

where X and U are the physical position and slope, and $\zeta = \beta^{1/2}$ and $\alpha = -2\beta'$. Each non-linear element will be labelled from 0 to N-1. Then the linear transfer matrix for one transverse plane for the passage from m to n is just

$$R(n-m) = \begin{pmatrix} \cos(\phi_{nm}), \sin(\phi_{nm}) \\ -\sin(\phi_{nm}), \cos(\phi_{nm}) \end{pmatrix} \tag{2}$$

where ϕ_{nm} is the betatron phase advance through that section. A non-linear element, n, will be described by

$$x_i(\text{out}) = x_i(\text{in}) + T(n)_{ijk} x_j(\text{in}) x_k(\text{in}) \ . \tag{3}$$

The matrix elements $T(n)_{ijk}$ are symmetric in the last two indices.

After passing through N cells, beginning at the face of the 0 non-linear element and ending at the face of the N non-linear element, the phase space vector $x_i(N)$ can be written as a power series of the initial conditions, $x_i(0)$

$$x_i(N) = \sum_{n=1}^{2N} T^{(n)}_{ijk\ldots l} x_i(0) x_j(0) x_k(0) \ldots x_l(0) \ . \tag{4}$$

The coefficients $T^{(n)}_{ijk\ldots l}$ are symmetric in all but the first index. The first few coefficients are listed below

$$T^{(1)}_{ij} = R_{ij}(N) \tag{5a}$$

$$T^{(2)}_{ijk} = \sum_{n=0}^{N-1} R_{i\alpha}(N-n) T(n)_{\alpha\beta\nu} R_{\beta j}(n) R_{\nu k}(n) \tag{5b}$$

$$\begin{aligned} T^{(3)}_{ijkl} = 2 \sum_{n_2=1}^{N-1} \sum_{n_1=0}^{n_2-1} & R_{i\alpha}(N-n_2) T(n_2)_{\alpha\beta\nu} R_{\beta\bar{j}}(n_2) \\ & \times R_{\nu\delta}(n_2-n_1) T(n_1)_{\delta\epsilon\mu} R_{\epsilon\bar{k}}(n_1) R_{\mu\bar{l}}(n_1) \\ & \times (\delta_{\bar{j}j}\delta_{\bar{k}k}\delta_{\bar{l}l} + \delta_{\bar{j}k}\delta_{\bar{k}j}\delta_{\bar{l}l} + \delta_{\bar{j}l}\delta_{\bar{k}k}\delta_{\bar{l}j})/3 \ . \end{aligned} \tag{5c}$$

In the above form, eqs. (5) are sufficiently general to describe a lattice of arbitrary complexity. They can be regarded as a logical elaboration of standard lattice theory going beyond the lowest non-linear order of approximation.

One Sextupole Family

The one sextupole family lattice we will use has sextupoles of strength m (in normalized units) separated in betatron phase by an amount, Δ. The matrix element for such a sextupole is

$$T_{211} = mx^2, \text{ otherwise } T_{ijk} = 0 \ . \tag{6}$$

Moreover, the value of m can be scaled into the variable x_i by $x_i \rightarrow x_i/m$. Any non-zero m will be scaled into the x_i this way and henceforward $m = 1$.

For ease of presentation, $x_i(0) = (x_0, 0)$. Thus (4) becomes

$$x_i(N) = \sum_{n=1}^{2N} T^{(n)}_i x_0^n \quad . \tag{7}$$

The first four coefficients have been evaluated using REDUCE. As an example, the $T^{(3)}_2$ coefficient, before simplification, is

$$T^{(3)}_2 = 2 \sum_{n_2=1}^{N-1} \sum_{n_1=0}^{n_2-1} \frac{\cos(\Phi - n_2\Delta)\cos(n_2\Delta)\sin((n_2-n_1)\Delta)}{\cos^2(n_1\Delta)} \tag{8}$$

Here Φ is the total phase advance through the N cells. First REDUCE can be made to expand products of trig functions appearing in each summation into a series of their Fourier components. Then it can perform the indicated summations by making appropriate substitutions. Sample substitutions are listed in the appendix. Finally, REDUCE can give the i=1 coefficients by taking a derivative with respect to Φ and multiplying by -1 each i=2 coefficient. The results of these manipulations are fairly long and will not be listed. Table I lists a comparison of the $x(N)$ found with the coefficients up to and including $T^{(4)}$ and tracking for various N's and Δ.

The discrepancy between the analytic solution and the tracking solution gets worse for larger N because higher order coefficients have a strong N dependence. This first occurs for $T^{(3)}_2$ which has a piece that depends linearly on N as

$$(N-1)\cos(\Phi)\cos(\Delta/2)(1+4\cos(\Delta))/16/\sin(3\Delta/2) \ . \tag{9}$$

Such a term arises because after a summation over one index terms can remain which are independent of other indices. As in (9), after the n_1 summation is done there will be a piece left with no n_2 dependence. Hence, when the n_2 sum is done we get the factor N-1. This cancellation behavior persists in higher orders also. $T^{(4)} = O(N)$, $T^{(5)} = O(N^2)$, etc. Thus, even though the exact N dependence of $T^{(M)}$ for M large has

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Table I

Comparison of x_{track} and x_{4th} with $x_0 = .1$, $u_0 = 0$

N	x_{track}	x_{4th}
Δ=0.340460692π		
3	-9.6135D-02	-9.6144D-02
5	5.7579D-02	5.7606D-02
10	-3.1054D-02	-3.1037D-02
50	-9.5195D-02	-9.6497D-02
100	9.9098D-02	1.0206D-01
1000	1.1435D-03	3.1065D-01
Δ=0.790583216π		
3	4.3466D-02	4.3461D-02
5	9.7897D-02	9.7901D-02
10	9.3357D-02	9.3371D-02
50	-4.2973D-01	-4.2875D-01
100	-9.5700D-02	-9.6685D-02
1000	9.6744D-02	1.2158D-01

not been determined, it is reasonable to rewrite the series (7) as

$$x_i(N) = \sum_{k=0}^{L} b_k(x_0, i) N^k \tag{10}$$

where L depends on N. It is to be understood that $b_k(x_0, i)$, $k>0$, contains all such offending terms. For example, b_1 will contain (9).

With the aim of estimating the "dynamic aperture" a modified Courant-Snyder "invariant", $\varepsilon(N)$, is defined by

$$\varepsilon(N) = x^2(N) + u^2(N) \quad .$$

Naturally, the non-linear elements cause ε to be not invariant, as Fig. 1 shows. In fact, for large enough initial conditions, $\varepsilon(N)$ will grow uncontrollably with increasing N. Our goal is to find a way to estimate the largest starting point, x_0, such that $\varepsilon(N)$, though it may fluctuate, will remain stable. This point is defined to be the dynamic aperture. One method of finding the dynamic aperture is particle tracking. Another is to use the truncation scheme described below.

In the stable region, as N becomes large the b_k's, for $k>0$, must become very small or else (10) would blow up like some power of N. This leads to the asymptotic argument that a truncated version $x^T{}_i(N)$, consisting only of the zeroth term in (10), would be a qualitatively good approximation to $x_i(N)$. Comparing an ε formed with $x^T{}_i$ and ε from tracking, as in Fig. 1, it can be seen that in the stable region this is a valid assumption; the average values of ε for both cases are quite close. After truncation, there is no possibility of uncontrolled growth in ε^T, since it is a finite Fourier series. Nevertheless, a kind of beating occurs which causes ε^T to oscillate over a progressively larger range as the limit of stability is exceeded. This tendency can be exploited to give a quantitative estimate.

The maximum value $\varepsilon^T{}_{max}$ of ε^T grows very fast as the starting point exceeds the dynamic aperture. The growth rate can be quantified by a factor K by which $\varepsilon^T{}_{max}$ is larger than its initial value.

$$\varepsilon^T{}_{max} = K\varepsilon(0) = Kx_0{}^2 \quad . \tag{11}$$

Empirically K increases rapidly from a nominal value of about 2 near the dynamic aperture determined from tracking. Turning this around the solution of (11) with K=2 yields an estimate of the dynamic aperture from the truncated behavior as shown in Fig. 2. This prediction depends on the arbitrarily chosen value of K but the sensitivity is slight and decreases as higher orders are included. For example, increasing K by a factor of 100 increases the estimate by a factor of perhaps 3.

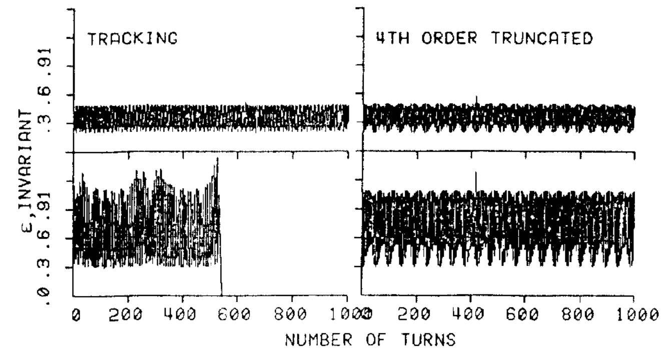

Fig. 1 ε from tracking and truncation
top $\varepsilon(0)=.36$, bottom $\varepsilon(0)=.64$, $\Delta=0.340460692\pi$

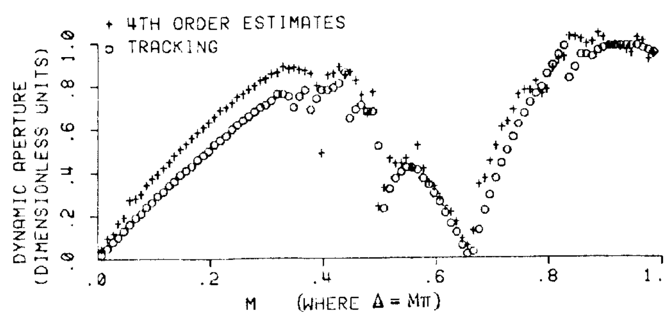

Fig. 2 Dynamic aperture for one sextupole family, m=1

Finally a word about the resonances shown in Fig. 2. If the analytic guess was done with only up to second order, i.e., only $T^{(2)}$ non-zero, then only the third integer is present. To third order the resonance at $2\pi/4$ appears. The $2\pi/5$ resonance comes from the fourth order terms.

Two or More Sextupole Families

Suppose there are N_f families of sextupoles. They are arranged sequentially as ABC...EABC...E... . Like sextupoles are Δ apart in phase. Between the different ones in a cell the separation in phase is defined to be $A \to B$ Δ_1, $A \to C$ Δ_2, $A \to D$ $\Delta_3, \ldots$, $A \to E$ Δ_{Nf-1}. This calculation is done only up to third order. Once again the strength of the initial sextupole has been scaled into the variable x_i, to make $m_A = 1$.

Specializing to two families, a practical case is when $m_B = -2$, which corresponds to a chromatically compensated FODO lattice[1] whose eta functions are in the ratio 2:1. There are three cases presented in Fig. 3. The factor P is the percentage of the total phase advance per cell from A to B, that is $\Delta_1 = P\Delta$. There seems to be some advantage to having most of the phase advance after sextupole B.

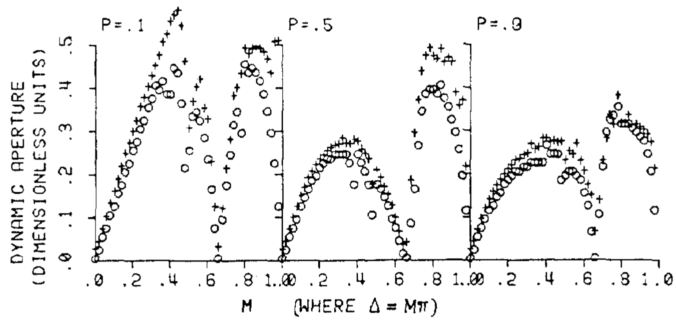

Fig. 3 Dynamic aperture for two sextupole families $\Delta_1 = P\Delta$ (o tracking, + 3rd order prediction)

Insertions With One Sextupole Family

An insertion (rf straight, mini-β, etc.) is a linear region characterized by the phase advance, ϕ, through it. One "turn" will consist of N cells and a linear insertion. A cell is a sextupole and a linear part with phase advance, Δ. A value of ε^T_{max} is obtained as above. Unfortunately, the predictions are done to second order only because no unambiguous way has been found to truncate the third order term. However, as Fig. 4 shows, second order predictions and tracking agree qualitatively. In this figure the numbers plotted specify ϕ as a multiple of $\pi/10$. Although the results are choppy, the trend indicates that $\phi = 0$ is an envelope for the $\phi \neq 0$ cases. Thus with ϕ being a multiple of 2π, the aperture seems to be its largest.

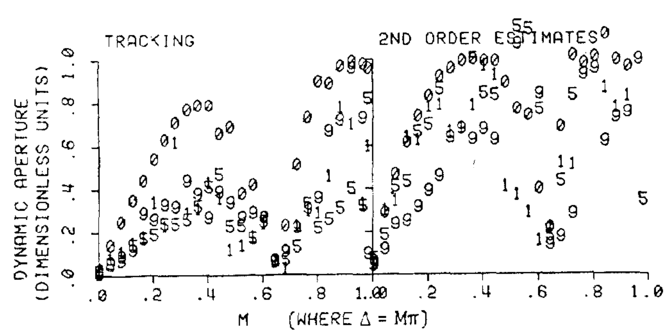

Fig. 4 Dynamic aperture over a range of Δ's for several insertions phases, $\phi = \#(\pi/10)$.

Conclusion

Using the symbolic algebraic manipulation program REDUCE, a prescription has been given for estimating the dynamic aperture of a non-linear lattice from the leading terms of a power series expansion. Reasonable agreement with apertures determined by tracking has been obtained with more detail in the form of resonant features being observable as the order of truncation is increased. Tentative recommendations for workable tunes and phase advances across insertions can be inferred from the figures.

Appendix

A sample summation substitution:

$$\sum_{n=k}^{N-1} \cos(x+ny) = \cos(x+(N+k-1)y/2)\,\frac{\sin((N-k)y/2)}{\sin(y/2)} \qquad (12)^2$$

$$\sum_{n_2=1}^{N-1} \sum_{n_1=0}^{n_2-1} \cos(x+n_1y+n_2z) = -\Big\{\sin(x-y/2+Nz/2) \times \frac{\sin((N-1)z/2)}{\sin(z/2)} - \sin(x-y/2+N(y+z)/2) \times \frac{\sin((N-1)(y+z)/2)}{\sin((y+z)/2)}\Big\}/2/\sin(y/2) \qquad (13)$$

References

1. K. Steffen, Simplified Formulae for Properties and Scaling of Large Electron Storage Rings, Internal Report, DESY PET-79/03.

2. 1.341.2 Gradshteyn and Ryzhik, Table of Integrals, Series and Product (Academic Press, New York, 1965).