A Theoretical Model and Computer Simulations to Describe Diffusion Enhancement by the Beam-Beam Interaction

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Abstract

Several observers have speculated[1,2,3] that the simultaneous presence of diffusion processes and the beam-beam interaction may lead to enhanced diffusion or beam loss greater than that present with either diffusion or the beam-beam interaction alone. To test those ideas we have developed analytical and numerical methods to estimate the magnitude and variation of this enhancement. We find substantial agreement between simulation results and our theoretical model that describes diffusion enhancement caused by a major resonance located within the beam. In this paper, we present results of beam-beam simulations with random diffusion which systematically cover fractional tunes between 0.0 and 0.5 and tune shifts between 0.0 and 0.10. Our theoretical model is tested systematically, and is found to provide an adequate approximate description of the phenomena.

Equations of Motion and Simulation Procedures

In the simulations below, hundreds of particle orbits are tracked through thousands of turns. Transport about one turn is simulated as the product of two matrix multiplications:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{Final}} = \begin{pmatrix} 1 & 0 \\ \frac{-4\pi\Delta\nu}{\beta} F(x) & 1 \end{pmatrix} \begin{pmatrix} \cos(2\pi\nu) & \beta\sin(2\pi\nu) \\ -\frac{1}{\beta}\sin(2\pi\nu) & \cos(2\pi\nu) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}. \quad (1)$$

We have approximated the beam-beam interaction as a "zero-length" - "weak-strong" interaction, where "zero-length" means a truncation of the interaction to a velocity kick and "weak-strong" means the force produced by the opposite "strong" beam is unchanged from turn to turn. The above transformation is equivalent to integration of the equation of motion

$$x'' + K(s)x = \frac{-4\pi\nu}{\beta_o} F(x)\, x\, \delta_p(s). \quad (2)$$

For F(x) we have used the 1-D truncation of the force from a round gaussian beam

$$F(x) = \frac{1-\exp(-x^2/2\sigma^2)}{x^2/2\sigma^2} \quad (3)$$

The simulation conditions and the parameters $\sigma^2=(.0816)^2\,\text{mm}^2$ and $\beta_o=2\text{m}$ are chosen to agree with expected Tevatron $\bar{p}p$ collision conditions.[4]

A tune at small amplitudes ν_o and the matched small amplitude betatron amplitude β_o can be found by noting that as $x\to 0$, the beam-beam force becomes

$$\Delta x' = \frac{-4\pi\Delta\nu}{\beta_o}\, x \quad (4)$$

and the transformations of Equation 1 are now linear. ν_o and β_o are found by considering the linear transformation from the center of an interaction to the next, obtaining

$$\cos 2\pi\nu_o = \cos(2\pi\nu) - 2\pi\Delta\nu \sin(2\pi\nu) \quad (5)$$

$$\beta_o \sin 2\pi\nu_o = \beta \sin 2\pi\nu. \quad (6)$$

The parameter values for the simulations below are set by the following procedure: β_o is set equal to the value of 2m for all cases. An approximate for ν_o, ν^*, is set to a "bench mark" value (0., 0.05, 0.10, 0.45). The parameter $\Delta\nu$ is then set to one of the values (0.0, 0.005, 0.01...0.10) and ν is set by $\nu=\nu^* - \Delta\nu$. The precise value of ν_o is found from Equation 5, and Equation 6 is used to find the transfer matrix value of β.

In each simulation a set of initial particle positions is generated randomly within a bigaussian distribution and then transported through 200,000 turns, with a random diffusion kick on each turn.

Random diffusion is simulated by adding a random velocity on each turn

$$x' \to x' + \Delta R \quad (7)$$

where R is a random number between -1 and +1 and Δ is an amplitude parameter. In the simulations reported in this note Δ is chosen with a size that doubles rms beam size within a few hundred thousand turns. This is larger than the expected random diffusion from beam-gas scattering, RF noise and power supply ripple in the Tevatron. The larger size is deliberately chosen to display noticeable effects in reasonable computer times.

The diffusion expected from a given value of Δ can be calculted. The change in emittance ε is

$$\frac{d\varepsilon}{dt} = \frac{d}{dt} 3\left(\frac{\overline{x^2}}{\beta_o} + \beta_o \overline{x'^2}\right) = \beta_o \Delta^2 \equiv D_o \quad (8)$$

In the simulations the quantity

$$\varepsilon = 3\left(\frac{\overline{x^2}}{\beta_o} + \beta_o \overline{x'^2}\right)$$

is calculated for a set of particle trajectories every 2000 turns. A straight line least squares fit for $\varepsilon(t)$ is calculated to find a diffusion constant D which differs from D_o by the enhancement factor α_E:

$$D = \alpha_E D_o. \quad (9)$$

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Simulation Results

Fig. 1 shows the effect of increasing the beam-beam interaction strength when the diffusion processes are present. Weak beam-beam interactions do not effect the diffusion. But beam-beam interactions stronger than a threshold value can double the diffusion. The threshold occurs when $\Delta\nu$ includes a low order resonance.

Table 1 and Figure 2 show a complete set of enhancement values for fractional tunes from 0.0 to 0.5 and tune shifts from 0.0 to 0.10. Two hundred particles were tracked for 200,000 turns. Those results combined with Fig. 3 (see also references 5,6,7) can be summarized.

1. α_E can be substantially greater than 1 if a major low order resonance (1/4, 1/6, 1/8) is withi the tune spread ν to $\nu + \Delta\nu$. The greatest enhancement is found for the lowest order resonances. (α_E = 6.39 for ν_o^* = 0.30 and $\Delta\nu$ = 0.06 on Table 1.)

2. α_E is independent of D_o.

3. The development of the rms increase in beam site depends on the location of the resonance within the tune spread. If the resonant tune is near ν, the enhancement is due to a few particles kicked to large amplitudes. If not, the enhancement is distributed more uniformly.

Diffusion Enhancements and Betatron Function Mismatch

In these simulations, the transport parameters are chosen such that motion of small amplitude particles is approximated by the linear matrix with β-function β_o, and tune $\nu_o = \nu_o^*$. Very large amplitude particles would have motion corresponding to β, ν and these will not equal β_o, ν_o. Other particles would have effective values between β and β_o, ν and ν_o.

Our formula for unperturbed diffusion

$$D_o = \beta_o \langle \Delta\theta^2 \rangle$$

should be replaced by

$$\bar{D}_o = \langle \beta \rangle \, \langle \Delta\theta^2 \rangle \qquad (10)$$

where $\langle\beta\rangle$ is a mean β-function value. We expect that α_E would be given by

$$\alpha_E = \frac{\langle\beta\rangle}{\beta_o} \qquad (11)$$

In the simulations $\beta > \beta_o$ for $\nu_o > .25$ and $\beta < \beta_o$ for $\nu_o < .25$ (see Table II, Reference 5). As can be seen from Table I, values of α_E for $\nu_o > 0.25$ and $\Delta\nu$ relatively large show $\alpha_E < 1$ as expected and those with $\nu_o < .25$ show $\alpha_E > 1$. This distortion is particularly large for $\nu \to 0.0$ where $\langle\beta\rangle \gg \beta_o$ and for $\nu_o \to .5$ where $\langle\beta\rangle \ll \beta_o$. The magnitudes of α_E are in agreement with that expected from Equation 10, except for cases with large resonances.

Resonance Analytical Model of Diffusion Enhancement

We have previously presented a model[1] to describe this enhancement which we investigate more thoroughly in this note. In this model we note that particle motion is significantly distorted by resonances, leading to changes in prticle amplitudes I, where

$$I = \frac{x^2}{2\beta_o} + \beta_o \frac{x'^2}{2} \qquad (12)$$

Particle amplitudes which reach the lower boundary of a resonance, I_T by random diffusion can reach the much larger amplitude $I_T + 2\Delta$ by an infinitesimal diffusion kick. (Δ is the resonance half-width.) The resonance places the amplitudes I_T and $I_T + 2\Delta$ adjacent so that diffusion moves rms particle amplitudes by 2Δ when I_T is reached.

We have defined rms emittance by

$$\varepsilon = 3 \left\langle \frac{x^2}{2\beta_o} + \frac{\beta_o x'^2}{2} \right\rangle = 6 \langle I \rangle \qquad (13)$$

Diffusion enhancement by resonances modifies D_o by

$$\dot{\varepsilon} = D = D_o + \frac{\dot{N}}{N}\bigg|_{I_T} (12\Delta) \qquad (14)$$

where $\dot{N}/N$ is the rate at which particles reach the threshold I_T.

$\dot{N}/N$ can be estimated by considering the change in the particle distribuution caused by random diffusion. The initial distribution is gaussian:

$$f(I) \sim \exp(-I/I_o) \qquad (15)$$

To estimate particle flux at I_T we calculate the change in the number of particles with $I < I_T$, obtaining

$$\frac{\dot{N}}{N}\bigg|_{I_T} \simeq -\frac{d}{dt}\int_0^{I_T} f(I)\,dI = \frac{I_T \dot{I}_o}{I_o^2} \exp\left(-I_T/I_o\right) \qquad (16)$$

$\dot{I}_o/I_o$ is the same as $\dot{\varepsilon}_o/\varepsilon_o$. Equation (15) can be written as

$$\dot{\varepsilon} \simeq \dot{\varepsilon}_o \left(1 + \frac{12 I_T}{\varepsilon_o I_o} \Delta \exp\left[-I_T/I_o\right]\right) \qquad (17)$$

α_E can now be calculated provided I_T and Δ can be measured on the phase-space trajectory plots of Reference 7 or calculated in the single resonance model.[8] In the Appendix of Reference 8 we describe this calculation. The two methods are in good agreement. In Table 2 we compare measured enhancement factors with calculated values from this model.

In Table 2 enhancement factors for 1/4, 1/6 and 1/8 resonances are calculated and compared with simulation results. In these, the measured I_T calculated Δ are used. The calculated α_E are not corrected for the shift in β noted above. This shift increases α_E for $\nu<0.25$, $\Delta\nu$ large and decreases α_E for $\nu>0.25$, $\Delta\nu$ large. The calculated values have included only the lowest order resonances and are very sensitive to small errors in the threshold I_T. The simulation values are subject to statistical fluctuations in the small subset of particle orbits which are in the resonance region. Calculated and simulation α_E agree to within ~10 to 20%, as shown in Table 2. This is within expected errors and shows that our theoretical model describes the difussion enhancement and can predict its magnitude.

References

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Figure 1 Beam emittance E as a function of time n_T with diffusion. We have set ν_0 = 0.2 and included cases with $\Delta\nu$ = 0., 0.02, 0.05, 0.10.

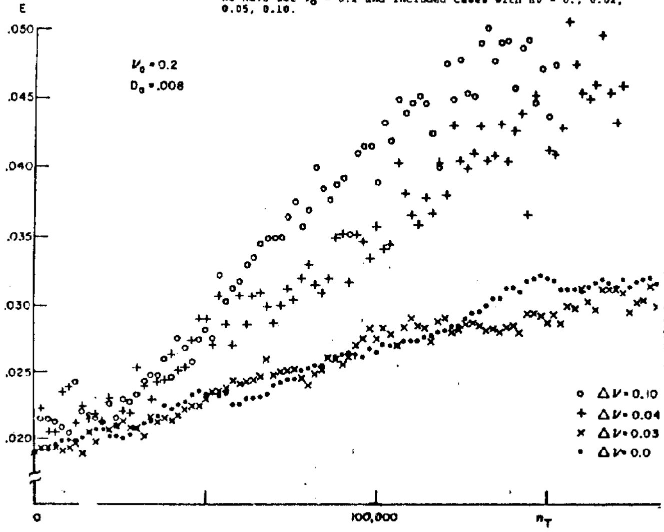

TABLE 1 Single precision enhancement values after 200,000 turns; 500 particles.

Δν										
0.100	6.02	1.47	1.81	2.80	2.52	1.00	0.60			
0.095	2.59	1.52	1.83	2.38	2.70	0.92	0.74			
0.090	2.74	1.17	1.91	2.11	2.79	0.90	0.81			
0.085	1.77	1.46	2.10	1.40	2.94	1.05	0.76			
0.080	1.75	1.50	1.65	1.19	3.55	0.97	0.70			
0.075	1.41	1.58	1.41	1.16	3.73	1.02	0.75			
0.070	1.37	1.54	1.37	0.93	3.80	1.01	0.83			
0.065	1.28	1.16	1.34	1.08	4.97	0.93	0.72			
0.060	1.32	1.62	1.54	1.10	6.92	0.74	0.76			
0.055	1.34	1.28	1.59	1.18	6.68	1.07	0.75			
0.050	4.08	1.18	1.17	1.90	1.13	1.33	0.90	0.96	0.42	
0.045	1.86	1.15	1.00	2.08	0.96	0.93	1.21	0.91	0.56	
0.040	1.70	1.12	1.19	2.04	1.10	0.97	1.21	0.96	0.56	
0.035	1.37	1.04	1.37	1.45	0.92	0.91	1.22	0.92	0.78	
0.030	1.26	1.18	1.39	1.18	1.14	0.94	1.41	1.01	0.70	
0.025	1.30	1.04	1.92	1.11	0.99	0.81	1.49	0.97	0.60	
0.020	1.16	1.40	1.09	1.05	1.00	0.96	1.59	0.94	0.81	
0.015	1.10	1.01	1.00	0.92	1.20	1.22	0.98	0.85	0.09	
0.010	1.09	1.03	0.96	1.01	1.09	1.12	0.90	1.01	0.80	
0.005	1.05	1.16	0.88	1.10	1.06	0.92	1.01	1.00	0.88	
0.000	0.93	0.82	0.98	1.25	0.99	1.20	1.05	0.98	0.81	0.98
ν_0 =	0.0001	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45

Table 2. Comparison of Simulation Diffusion Enhancement Results with Model Calculations.

Resonances included	ν	Δν	α_E Simulation	α_E Calculated
1/4	0.2	0.10	2.5	2.3
	0.205	0.095	2.7	2.6
	0.21	0.09	2.8	3.1
	0.215	0.085	2.9	3.4
	0.22	0.08	3.6	3.8
	0.225	0.075	3.7	4.6
	0.23	0.07	3.8	5.6
	0.235	0.065	5.0	6.1
	0.24	0.06	6.9	6.8
	0.245	0.055	6.7	6.2
1/6 , 1/8	0.15	0.10	2.8	2.2
	0.155	0.095	2.4	2.1
	0.10	0.10	1.8	1.5
	0.105	0.095	1.8	1.4
	0.11	0.09	1.8	1.6
	0.115	0.085	2.1	1.9
	0.12	0.08	1.6	1.3
1/6	0.125	0.075	1.4	1.4
	0.13	0.07	1.4	1.5
	0.135	0.065	1.3	1.5
	0.14	0.06	1.5	1.7
	0.145	0.055	1.6	1.8
	0.15	0.05	2.0	2.0
	0.155	0.045	2.1	2.2
	0.16	0.04	2.0	1.9
1/3	0.32	0.03	1.4	1.5
	0.325	0.025	1.5	1.6
	0.33	0.02	1.6	1.6

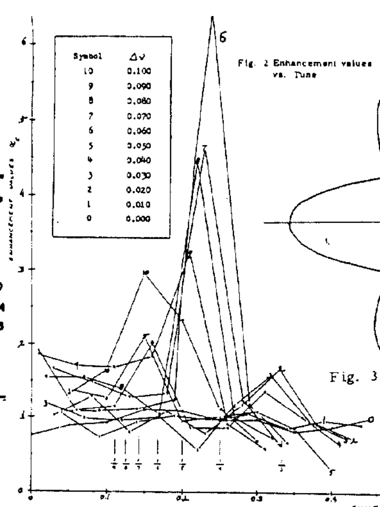

Fig. 2 Enhancement values vs. Tune

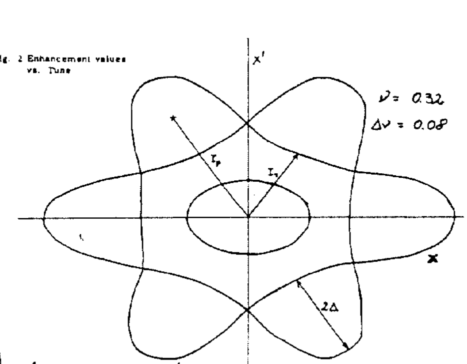

Fig. 3 (x, x')- Phase Space Plot