Searches for "Arnold Diffusion" and "Chaotic" Motion in the Beam-Beam Interaction*

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Abstract

The results of long time computer simulation searches for "Arnold Diffusion" in beam-beam interactions at $\bar{p}p$ collider parameters are presented. No evidence of significant diffusion is found, consistent with beam storage for many days without "blow-up". "Chaotic" motion is observed in 2-D simulations at $\bar{p}p$ parameters by use of reversability tests, and Lyapunov exponents for the chaotic motion are measured. This motion appears at the intersections of low-order resonances. Chaotic motion in 1-D simulations and in simulations with tune modulation can also occur. The conditions for the appearance of chaotic motion are explored as well as its relationship with "beam blow-up".

Introduction

In proton-antiproton ($p\bar{p}$) storage ring machines, p and $\bar{p}$ bunches collide with each other, providing high luminosity, high energy interactions. In these collisions particle motions in the colliding bunches are perturbed by the electromagnetic field of the opposite bunch, which provides the nonlinear "beam-beam" interaction. This nonlinear force can cause beam loss or "blow-up" reducing luminosity. $\bar{p}p$ colliders are particularly vulnerable to this instability since the difficulty of accumulating $\bar{p}$'s necessitates that the beams must be stored for long periods (~ 1 day) without significant beam "blow-up" in the collider.

Our simulations approximate particle motion in the "Tevatron"[2], and track millions of turns of particle motion around the ring with a nonlinear beam-beam "kick" on each turn.

Motion around the ring from "kick" to "kick" is simulated by multiplying particle coordinates (x, x´, y, y´) by a linear 4x4 "Courant-Snyder" matrix[3] determined by the C-S parameters: ν_x, ν_y ("tune"); β_x, β_y, α_x, α_y. x and y motion are uncoupled in the matrix and $\alpha_x=\alpha_y=0$ and $\beta_x=\beta_y=2$ m are chosen to fit "Tevatron" values. Note that only the fractional parts of the C-S tunes are significant in the matrix; integer parts are therefore ignored in this paper.

The beam-beam interaction is simulated by adding a nonlinear kick to the velocity (x , y) of the form $x' \rightarrow x'+F_x(x,y)$, where

$$F_x = \frac{-4\pi\,\Delta\nu}{\beta_o}\,\frac{(1 - e^{-(x^2+y^2)/2\sigma^2})x}{(x^2+y^2)/2\sigma^2}$$

$\Delta\nu$ is the "linear tune shift", β_o is the C-S parameter and the function is the beam-beam "kick" due to a round gaussian "strong" beam with rms radius σ (σ=.0816 mm in the Tevatron). The function is unchanged throughout the simulations, which approximates collisions of a weak $\bar{p}$-beam with an unaffected "strong" p-beam. The parameters of the simulations closely approximate conditions in a $\bar{p}$-p collider: $\Delta\nu \leq .01$ "zero-length", "weak-strong" 2D collisions of round beams, with no synchrotron radiation effects.

Long Time Simulation Search for "Arnold Diffusion"

Nonlinear dynamics research has noted that 2D motion in a nonlinear field can develop excursions to large amplitude at large time scales, by a process such as "Arnold Diffusion" in which particle trajectories follow stochastic motion along many intersecting resonances[1]. To determine whether such motion occurs at $\bar{p}p$ collider parameters we have undertaken long time scale simulations with high accuracy and searched for signs of instability both in RMS beam size and in individual trajectories.

High accuracy is useful in separating an intrinsic instability of the nonlinear dynamics from the noise-like effects of inadequate accuracy. We use CDC double precision to obtain 28 decimal place accuracy in a single turn calculation. Long time simulation accuracy is checked by following individual particle trajectory forward in time for many turns and then reversing the motion, comparing forward and return particle coordinates. Figure 1 shows the results of such "reversability tests" for sample trajectories of case B (see below) in 60 million turns forward and return (120 million total) tests. After 120 million turns, initial and final positions agree to approx. 14 decimal places. Similar results are obtained in all our simulations for normal "non-chaotic" trajectories (see below), and this indicates the scale of accumulated error in our simulations.

Long time simulations of up to 120 million turns have been obtained. This corresponds to about 40 minutes of Tevatron beam storage (20μs period), a significant fraction of the $\bar{p}p$ storage cycle (several hours). In these cases the trajectories from 100 randomly chosen initial conditions are followed and RMS emittances ε_x and ε_y as a function of time are calculated. Fits of these values to straight lines are used to extrapolate "doubling times" for ε_x, ε_y.

Three cases have been explored in this long time mode, as described in more detail in references[4]:

Case A (ν_x=.245, ν_y=.245, $\Delta\nu$=.01):

This is a case with $\nu_x=\nu_y$ with tune spread covering the resonance ν=1/4. After 120 million turns no statistically significant changes in RMS emittances are observable with extrapolated "doubling time" of of ~200 days corresponding to changes of emittance of ≤0.01% over the full simulation time. The motion is completely stable both in individual particle trajectories and in the beam as a whole.

The spherical symmetry ($\nu_x=\nu_y$, etc.) of case A implies that the quantity $p_\theta=x'y-y'x$ is a constant of motion and the equations of motion[5] can be reduced to 1D; therefore, no multidimensional instability can be expected. This agrees with the stability of the simulations, in spite of the low order resonances.

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Case B (ν_x=.245, ν_y=.12, $\Delta\nu$=.01): This case has $\nu_x \neq \nu_y$ so that motion is truly 2D (unlike case A) and low order (1/4, 1/8) resonances within the tune spread. A 120 million turn simulation with 100 particle trajectories has been completed in this case. No large scale instability is observable and RMS emittance changes of ≤1% are observed, with extrapolated doubling times of ≥ several days.

However a systematic exchange of emittance between x and y motion of approx. 1% is observable and this is associated with the appearance of "chaotic" motion in some trajectories (see below). This "chaotic" motion does not lead to significant increases in RMS beam sizes, and does not lead trajectories to larger amplitudes at $\bar{p}p$ time scales.

Case C (ν_x=.3439, ν_y=.1772, $\Delta\nu$=.01): This is a case with $\nu_x \neq \nu_y$ and a tune spread free of resonances lower than ninth order. A 60 million turn simulation shows RMS emittance changes of ≤ .02% with extrapolated doubling times of ≥ 100 days. No "chaotic" motion is observed and the case shows complete stability.

We summarize the above simulations in the conclusion that the beam-beam interaction at $\bar{p}p$ colliders contains no intrinsic instability endangering useful luminosity within many hours of beam storage.

The Appearance of Chaotic Motion in the Beam-Beam Interaction

Repeatability tests of some trajectories in case B show substantially different behavior from those of figure 1, and a typical case is shown in figure 2. These trajectories, which we label "chaotic", develop errors of order unity in ~30-100 thousand turns. The other "normal" trajectories have errors of order 10^{-15} after 100 million turns.

In "normal" trajectories, errors δ grow as simple powers of the number of turns N

$$\delta \cong \delta_o N^{\alpha}$$

where δ_o is a single turn error size (10^{-28}) and α < 2 in our simulations. "Chaotic" trajectory errors grow exponentially

$$\delta \cong \delta_o e^{aN}$$

where, in case (B), a $\cong 10^{-3}$ is found empirically.

The reversability test was applied to 500 trajectories with initial conditions randomly chosen in the 4D gaussian phase space determined by σ. 75% are "normal" with small errors; 25% show exponential error growth with exponents a_i, with $\bar{a}_i \cong$ .001; there are no intermediate cases. These "chaotic" trajectories are associated with the phase space region affected by the 1/4, 1/8 resonances.

The results can be correlated with the developing concepts of nonlinear dynamics . The coefficient a_i is identified as the "Lyapunov Exponent" of the trajectory and its non-zero values indicate that the transformation has non-zero "Kolmogorov Entropy", which is found by averaging a_i over the phase space of the transformation T:

$$h(T) = 0.75\cdot 0 + 0.25\cdot a \cong 2.5\times10^{-4}$$

This entropy is significantly non-zero and in fact quite large in view of the "non-chaotic" appearance of the transformation.

Dependence of Chaotic Motion on Tunes (ν_x,ν_y) and Tune Spread ν

While 25% of trajectories are chaotic in case B, case A and C show no chaotic motion at all, which implies <1% chaotic motion. We have undertaken a systematic study of the dependence of chaotic motion on "tunes" and "tune spread" (ν_x,ν_y,$\Delta\nu$). We find that chaotic motion occurs when the tune spread contains the intersection of low order resonances. A resonance is determined by a relationship between the tunes:

$$n\,\nu_x + m\,\nu_y = p$$

where n, m and p are integers. The order "Ω" of the resonance is given by

$$\Omega = |n| + |m| \text{ if both n and m are even or}$$
$$\Omega = 2(|n|+|m|) \text{ if m and/or n are odd}$$

for our round beam-beam force.

The dependence of chaotic motion on resonance intersection is shown in figure 3. In each case the resonance intersection is placed in the center of the tune spread and 100 trajectories are tested for chaotic motion by a repeatability test. The results can be summarized:

1. Intersections of 4th and 6th order resonances show large regions of chaotic motion (10-30%).
2. Intersections of 4th or 6th order with 8th order also show some chaotic motion (≤5%).
3. Higher order intersections show little or no chaotic motion (≤1%).
4. Cases with ν_x=ν_y (or ν_x=ν_y±.5) show no chaotic motion. This is probably due to a kinematic invariant (p_θ) associated with the symmetry of the motion which reduces the motion to 1D as in case A above[5].

Dependence on tune shift $\Delta\nu$ has also been explored with $\Delta\nu$=.005, .01, .02. In these cases, the density of chaotic motion shows no dependence on tune shift. However the Lyapunov exponents a_i are directly proportional to $\Delta\nu$.

The results correlate well with resonance theory since for the beam-beam interaction the resonance width does not depend on $\Delta\nu$ but the frequency of oscillations within the resonance is directly proportional to $\Delta\nu$. We can thus correlate the density of chaotic motion with the area of resonance overlap and the Lyapunov exponents with unstable motion due to overlapping oscillations. We also note here that "chaotic" motion does not imply long-time instability. Our studies of case B show long-time stability even with large degrees of chaotic motion. We speculate that this limitation in amplitude changes is due to our "round" beam-beam force; "flat" beams may show greater amplitude changes.

We note that the repeatability test has been a very useful tool in separating "chaotic" from non-chaotic motion in beam-beam simulations and has done this with very little ambiguity.

Chaotic Motion in 1D Simulations

In the above 2D simulations, we see chaotic motion at the crossings of separate resonances. In 1D motion such crossings do not occur. We have explored conditions for chaotic motion for 1D cases with reversability tests. in which 1D motion is obtained by setting y=y′ identically to zero in the transformations described above.

In 1D motion large chaotic regions (≳ 10%) are found at large values of tune shift ($\Delta\nu \gtrsim .15$) where the motion can contain many low order resonances. For smaller tune shifts chaotic motion can occur near the separatrices of low order resonances but the total area occupied by the chaotic region is quite small.

In figure 4 we show phase space trajectories in a case ($\nu = .16$, $\Delta\nu = .09$) where interference between a sixth order resonance and a higher order resonance can be seen in the motion. Chaotic motion does occur near the separatrix of the sixth order resonance, but this occupies only a small fraction of the total phase space. Chaotic motion with smaller $\Delta\nu$ is confined to much smaller areas even with low order resonances.

Chaotic Motion with Tune Modulation

The tunes of the C-S transformation can be changed from turn to turn following

$$\nu_x = \nu_{xo} + a_x \sin \frac{2\pi}{N_x}$$

where a_x, N_x are the amplitude and period of the modulation. This can simulate modulations from power supply ripple and uncorrected chromaticity, etc. As we describe in a separate paper[6], modulations of this type can lead to chaotic motion in both 2D and 1D simulations if the modulation crosses a low order resonance. Unlike our 2D cases, this modulation can lead to amplitude instabity and beam blow-up.

References

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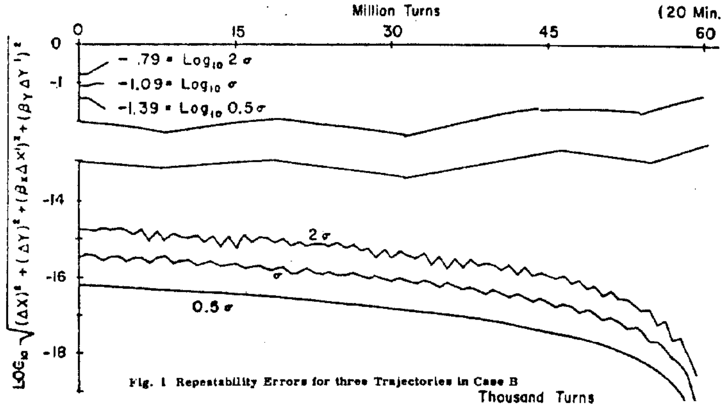


Fig. 1 Repeatability Errors for three Trajectories in Case B

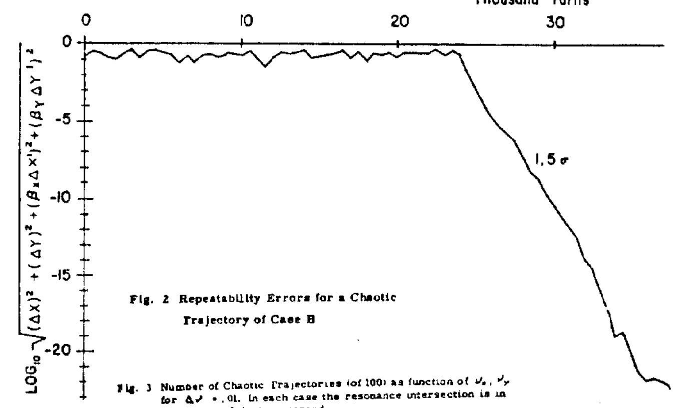
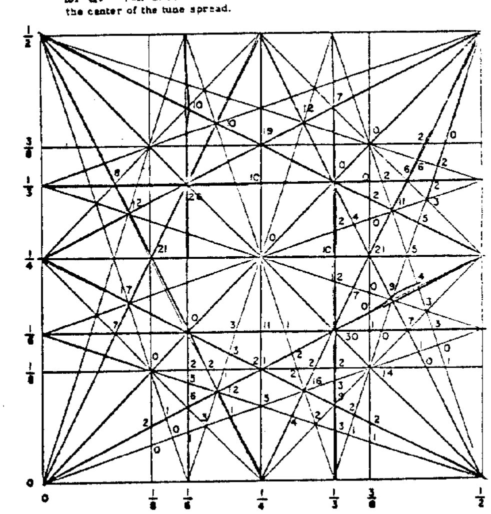


Fig. 2 Repeatability Errors for a Chaotic Trajectory of Case B

Fig. 3 Number of Chaotic Trajectories (of 100) as function of ν_x, ν_y for $\Delta\nu$ = .01. In each case the resonance intersection is in the center of the tune spread.



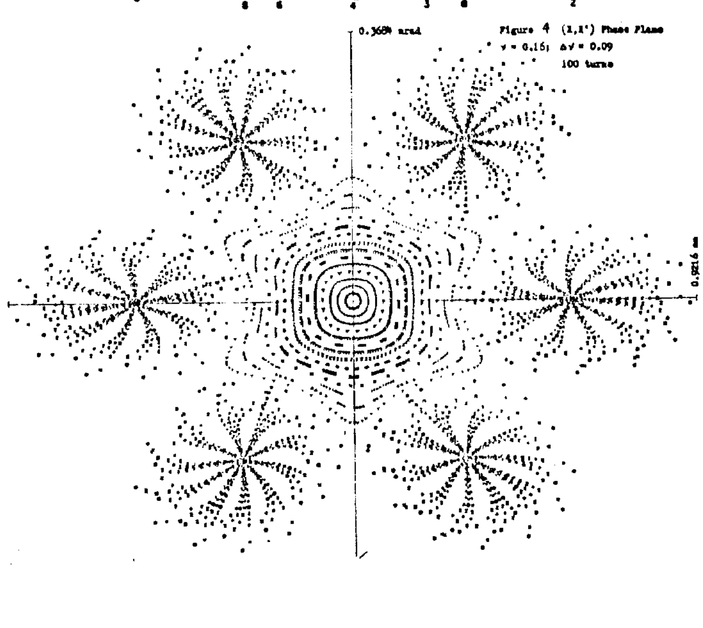


Figure 4 (x,x′) Phase Plane ν = 0.16; $\Delta\nu$ = 0.09 100 turns